INSTRUCTOR: RUIWEN SHU

Total time: 75 minutes.
Total points: 100.

Problem 1 (20 points). Find the general solution to the following system of differential equations.

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = 4y_1 - 2y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_1 + 2y_2 \end{cases}$$

This is $\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = A\mathbf{Y}$ with $A = \begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}$. First calculate eigenvalues:

$$(4-\lambda)(2-\lambda) - (-2)\cdot 1 = 0$$

$$\lambda^2 - 6\lambda + 10 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$$

Take the eigenvalue $\lambda = 3 + i$, calculate eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 4-(3+i) \end{pmatrix} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

Then a complex solution is

$$\mathbf{Y} = e^{(3+i)t}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = e^{3t}(\cos t + i\sin t)\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = e^{3t}\begin{pmatrix} 2\cos t + 2i\sin t \\ \cos t + i\sin t - i\cos t + \sin t \end{pmatrix}$$

$$= e^{3t}\begin{pmatrix} 2\cos t \\ \cos t + \sin t \end{pmatrix} + ie^{3t}\begin{pmatrix} 2\sin t \\ \sin t - \cos t \end{pmatrix}$$

Taking real / imaginary parts as special solutions, we get the general solution as

$$\mathbf{Y} = C_1 e^{3t}\begin{pmatrix} 2\cos t \\ \cos t + \sin t \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2\sin t \\ \sin t - \cos t \end{pmatrix}$$

Problem 2 (20 points). Solve the following initial value problem.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ 6 & 2 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

First calculate eigenvalues:

$$(-3 - \lambda)(2 - \lambda) - 1 \cdot 6 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda_1 = -4, \quad \lambda_2 = 3$$

The corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

Therefore the general solution is

$$\mathbf{Y} = C_1 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

Using the initial condition, we get

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

that is,

$$-C_1 - C_2 = 3, \quad C_1 - 6C_2 = -1$$

Solve to get

$$C_1 = -\frac{19}{7}, \quad C_2 = -\frac{2}{7}$$

Therefore the solution to initial value problem is

$$\mathbf{Y} = -\frac{19}{7} e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{2}{7} e^{3t} \begin{pmatrix} -1 \\ -6 \end{pmatrix}$$

Problem 3 (20 points). Sketch the phase portrait of the following system of differential equations.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \mathbf{Y}$$

First calculate eigenvalues:

$$(1-\lambda)(-3-\lambda) - 4 \cdot (-2) = 0$$

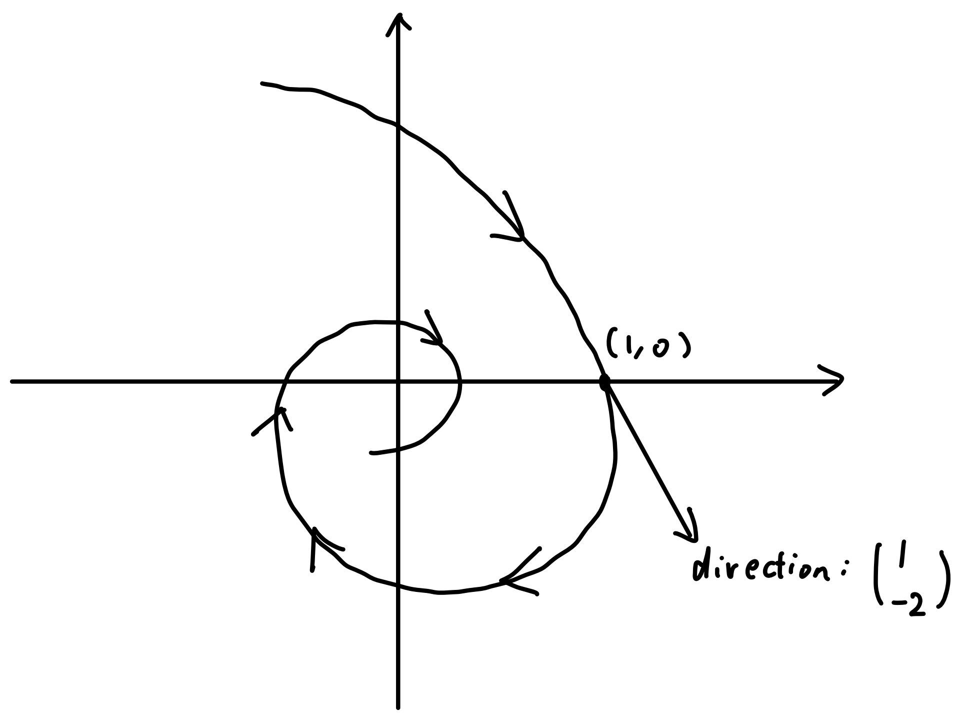
$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

Therefore we have a spiral sink. To determine its rotation direction, we check the vector field at $(1, 0)$:

$$\begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which points into the fourth quadrant. Therefore we get the following picture:



Problem 4 (20 points). Find the general solution to the following differential equation.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 8y = 3e^{2t}$$

First calculate eigenvalues:

$$s^2 + 6s + 8 = 0$$
$$(s+2)(s+4) = 0$$
$$s_1 = -2, \quad s_2 = -4$$

Therefore the right hand side exponent, 2, is not an eigenvalue. Try $y(t) = Ce^{2t}$:

$$4Ce^{2t} + 6 \cdot 2Ce^{2t} + 8 \cdot Ce^{2t} = 3e^{2t}$$

$$24C = 3$$
$$C = \frac{1}{8}$$

Therefore we get a particular solution

$$y(t) = \frac{1}{8}e^{2t}$$

Therefore the general solution is

$$y(t) = \frac{1}{8}e^{2t} + C_1 e^{-2t} + C_2 e^{-4t}$$

Problem 5 (15 + 5 = 20 points). Consider the following differential equation.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = \sin(2t) \quad \cdots (1)$$

(1) Find the general solution.
First consider the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = e^{2it} \quad \cdots (2)$$

First calculate eigenvalues:

$$s^2 + 4 = 0$$

$$s = \pm 2i$$

Therefore the right hand side exponent, $2i$, is a single eigenvalue. Try $y(t) = Cte^{2it}$:

$$C(4ie^{2it} - 4te^{2it}) + 4Cte^{2it} = e^{2it}$$

$$4iC = 1$$

$$C = \frac{1}{4i} = -\frac{1}{4}i$$

Therefore we get a particular solution to equation (2):

$$y(t) = -\frac{1}{4}ite^{2it} = -\frac{1}{4}it(\cos 2t + i\sin 2t) = \frac{1}{4}t\sin 2t - i\frac{1}{4}t\cos 2t$$

Taking imaginary part, we get a particular solution to equation (1):

$$y(t) = -\frac{1}{4}t\cos 2t$$

Therefore the general solution to equation (1) is

$$y(t) = -\frac{1}{4}t\cos 2t + C_1\cos 2t + C_2\sin 2t$$

(2) Describe the qualitative behavior of a solution to this differential equation as t gets large.

As t gets large, a solution oscillates with larger and larger amplitude, showing a resonance phenomenon.