Ex $\quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = -2e^{-t}, \quad y(0) = 0, \quad y'(0) = -1$

Solve by Laplace.

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + 4(s\mathcal{L}[y] - y(0)) + 3\mathcal{L}[y] = -2\cdot\frac{1}{s+1}$$

$$s^2\mathcal{L}[y] + 1 + 4(s\mathcal{L}[y]) + 3\mathcal{L}[y] = -2\cdot\frac{1}{s+1}$$

$$\underbrace{(s^2+4s+3)}_{=(s+1)(s+3)}\mathcal{L}[y] = -2\cdot\frac{1}{s+1} - 1$$

$$\mathcal{L}[y] = -2\frac{1}{(s+1)^2(s+3)} - \frac{1}{(s+1)(s+3)}$$

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s+1)$$

$s=-3 \Rightarrow 1 = B\cdot(-2) \qquad B = -\frac{1}{2}$

$s=-1 \Rightarrow 1 = A\cdot 2 \qquad A = \frac{1}{2}$

$$\frac{1}{(s+1)^2(s+3)} = \frac{C}{s+3} + \frac{D}{s+1} + \frac{E}{(s+1)^2}$$

$$1 = C\cdot(s+1)^2 + D\cdot(s+1)(s+3) + E\cdot(s+3)$$

$s=-3 \Rightarrow 1 = C\cdot(-2)^2 \qquad C = \frac{1}{4}$

$s=-1 \Rightarrow 1 = E\cdot 2 \qquad E = \frac{1}{2}$

s^2 coeff. $\Rightarrow 0 = C + D \qquad D = -\frac{1}{4}$

$$\mathcal{L}[y] = -2\left(\frac{1}{4}\cdot\frac{1}{s+3} - \frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{(s+1)^2}\right) - \left(\frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s+3}\right)$$

$$= -\frac{1}{(s+1)^2}$$

$$\boxed{\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}}$$

$$y = -te^{-t}$$

Ex $\quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = \delta_3(t), \quad y(0)=2, \quad y'(0)=0$

$$\boxed{\mathcal{L}[\delta_a] = e^{-as}}$$

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + 4(s\mathcal{L}[y] - y(0)) + 5\mathcal{L}[y] = e^{-3s}$$

$$s^2\mathcal{L}[y] \quad -2s \quad + 4(s\mathcal{L}[y] - 2 \quad) + 5\mathcal{L}[y] = e^{-3s}$$

$$(s^2 + 4s + 5)\mathcal{L}[y] = e^{-3s} + 2s + 8$$

$$\mathcal{L}[y] = e^{-3s}\frac{1}{(s+2)^2+1} + \frac{2s+8}{(s+2)^2+1}$$

$$\boxed{s^2+4s+5 = (s+2)^2+1}$$

$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$$

$$\mathcal{L}^{-1}[e^{-3s}\underbrace{\frac{1}{(s+2)^2+1}}_{F}] = u_3(t)\cdot e^{-2(t-3)}\sin(t-3)$$

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}$$

$$a=-2 \qquad \omega=1$$

$$2s+8 = \underline{2}(s+2) + \underline{4}\cdot 1$$

$$\mathcal{L}[e^{-2t}\cos t] = \frac{s+2}{(s+2)^2+1}$$

$$\mathcal{L}^{-1}\left[\frac{2s+8}{(s+2)^2+1}\right] = 2e^{-2t}\cos t + 4e^{-2t}\sin t$$

$$\mathcal{L}[e^{-2t}\sin t] = \frac{1}{(s+2)^2+1}$$

$$y = u_3(t)\cdot e^{-2(t-3)}\sin(t-3) + 2e^{-2t}\cos t + 4e^{-2t}\sin t$$

Ex $\quad \frac{dy}{dt} = \begin{cases} 2y + e^{-t} & 0 \le t < 1 \\ 2y + e^{-2t} & t \ge 1 \end{cases} \qquad y(0)=0$

$$= \underline{2y + e^{-t}} + u_1(t)\cdot\underline{(e^{-2t} - e^{-t})}$$

$$= 2y + e^{-t} + u_1(t)e^{-2t} - u_1(t)e^{-t}$$

$$= u_1(t)e^{-2(t-1)} \cdot e^{-2} \qquad = u_1(t) \cdot e^{-(t-1)} \cdot e^{-1}$$

$a=1 \quad f = e^{-2t} \quad F = \frac{1}{s+2} \qquad a=1 \quad f = e^{-t} \quad F = \frac{1}{s+1}$

$$s\mathcal{L}[y] - \cancel{y(0)} = 2\mathcal{L}[y] + \frac{1}{s+1} + e^{-2}e^{-s} \cdot \frac{1}{s+2} - e^{-1}e^{-s} \cdot \frac{1}{s+1}$$

$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$$

$$(s-2)\mathcal{L}[y] = \frac{1}{s+1} + e^{-2}e^{-s} \cdot \frac{1}{s+2} - e^{-1}e^{-s} \cdot \frac{1}{s+1}$$

$$\mathcal{L}[y] = \frac{1}{(s+1)(s-2)} + e^{-2}e^{-s} \cdot \frac{1}{(s+2)(s-2)} - e^{-1}e^{-s}\frac{1}{(s+1)(s-2)}$$

$$\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s+1)$$

$s=2 \Rightarrow 1 = B \cdot 3 \qquad B = \frac{1}{3}$

$s=-1 \Rightarrow 1 = A \cdot (-3) \qquad A = -\frac{1}{3}$

$$\frac{1}{(s+2)(s-2)} = \frac{C}{s+2} + \frac{D}{s-2}$$

$$1 = C(s-2) + D(s+2)$$

$s=2 \Rightarrow 1 = D \cdot 4 \qquad D = \frac{1}{4}$

$s=-2 \Rightarrow 1 = C(-4) \qquad C = -\frac{1}{4}$

$$\mathcal{L}[y] = -\frac{1}{3} \cdot \frac{1}{s+1} + \frac{1}{3} \cdot \frac{1}{s-2} + e^{-2}e^{-s}\left(-\frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{4} \cdot \frac{1}{s-2}\right)$$

$$- e^{-1}e^{-s}\left(-\frac{1}{3} \cdot \frac{1}{s+1} + \frac{1}{3} \cdot \frac{1}{s-2}\right)$$

$$y = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{4}e^{-2}u_1(t)e^{-2(t-1)} + \frac{1}{4}e^{-2}u_1(t)e^{2(t-1)}$$

$$+ \frac{1}{3}e^{-1}u_1(t)e^{-(t-1)} - \frac{1}{3}e^{-1}u_1(t)e^{2(t-1)}$$

$$\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$$

Ex Find general sol'n :

$$\frac{dy}{dt} + \frac{\cos 2t}{y} = 0$$

$$\frac{dy}{dt} = -\frac{\cos 2t}{y}$$

$$y\,dy = -\cos 2t\,dt$$

$$\int y\,dy = -\int \cos 2t\,dt$$

$$\frac{1}{2}y^2 = -\left(\frac{1}{2}\sin 2t + C\right)$$

$$y^2 = -(\sin 2t + 2C)$$

$$y = \pm\sqrt{-(\sin 2t + 2C)}$$

$$\int \cos 2t\,dt$$
$$= \frac{1}{2}\int \cos u\,du \qquad u = 2t \quad du = 2\,dt$$
$$= \frac{1}{2}\sin u + C$$
$$= \frac{1}{2}\sin 2t + C$$

Ex $\frac{dy}{dt} = \frac{y + t^2}{t} = \frac{y}{t} + \frac{t^2}{t} = \frac{1}{t}y + t$

$$\frac{dy}{dt} - \frac{1}{t}y = t$$

integrating factor $\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln(t^{-1})} = \frac{1}{t}$

$$\frac{d}{dt}\left(\frac{1}{t}y\right) = 1$$

$$\frac{1}{t}y = \int 1\,dt = t + C$$

$$y = t^2 + Ct$$

Ex $\frac{dy}{dt} = y^2 e^{-t}$ $\rightarrow$ missing sol'n $y = 0$

$$\frac{dy}{y^2} = e^{-t}\,dt$$

$$\int \frac{dy}{y^2} = \int e^{-t}\,dt$$

$$-\frac{1}{y} = -e^{-t} + C$$

$$\frac{1}{y} = e^{-t} - C$$

$$y = \frac{1}{e^{-t} - C}$$

$$\int e^{-t}\,dt \qquad u = -t,\quad du = -dt$$

$$= -\int e^{u}\,du$$

$$= -e^{u} + C$$

$$= -e^{-t} + C$$

$\Rightarrow$ general sol'n : $y = \dfrac{1}{e^{-t} - C}$ or $y = 0$.

Ex $\dfrac{dy}{dt} + \dfrac{2}{t}y = e^{t^3}$

$$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = t^2$$

$$\frac{d}{dt}(t^2 y) = t^2 e^{t^3}$$

$$t^2 y = \int t^2 e^{t^3}\,dt$$

$$= \frac{1}{3}e^{t^3} + C$$

$$y = \frac{1}{t^2}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$\int t^2 e^{t^3}\,dt \qquad u = t^3,\quad du = 3t^2\,dt$$

$$= \frac{1}{3}\int e^{u}\,du$$

$$= \frac{1}{3}e^{u} + C$$

$$= \frac{1}{3}e^{t^3} + C$$