6.4 (continued)

$$\delta_a(t) = \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}, \quad \int_0^\infty \delta_a(t)\,dt = 1$$

- If $\varphi(t)$ is a continuous function, $\int_0^\infty \varphi(t)\delta_a(t)\,dt = \varphi(a)$

$$\Rightarrow \boxed{\mathcal{L}[\delta_a]} = \boxed{\int_0^\infty \delta_a(t) e^{-st}\,dt = e^{-as}}$$

Ex $\quad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \delta_2(t), \quad y(0)=0, \; y'(0)=1$

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + 2(s\mathcal{L}[y] - y(0)) + 5\mathcal{L}[y] = e^{-2s}$$

$$s^2\mathcal{L}[y] - 1 + 2s\mathcal{L}[y] + 5\mathcal{L}[y] = e^{-2s}$$

$$(s^2+2s+5)\mathcal{L}[y] = 1 + e^{-2s}$$

$$s^2+2s+5 = (s+1)^2+4$$

$$\mathcal{L}[y] = \frac{1}{s^2+2s+5} + e^{-2s}\frac{1}{s^2+2s+5}$$

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}$$

$$\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}$$

$a=-1, \; \omega=2$

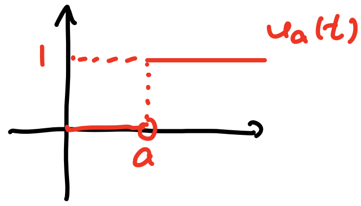
$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{1}{2}e^{-t}\sin 2t$$

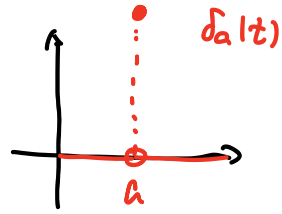
$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s^2+2s+5}\right] = u_2(t)\cdot\frac{1}{2}e^{-(t-2)}\sin(2(t-2))$$

$$\boxed{\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)}$$

$$y = \frac{1}{2}e^{-t}\sin 2t + u_2(t)\cdot\frac{1}{2}e^{-(t-2)}\sin(2(t-2))$$

Fact : $u_a'(t) = \delta_a(t)$



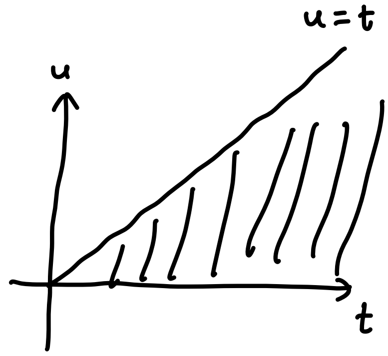


$$\int_0^t \delta_a(v)\,dv = \begin{cases} 0, & 0<t<a \\ 1, & t>a \end{cases}$$

6.5 Convolutions (NOT in exam).

Def Let $f(t)$, $g(t)$ be functions on $[0,\infty)$. The convolution of f and g is a function $f*g$ on $[0,\infty)$, defined by

$$(f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$$



$$\mathcal{L}[f*g] = \int_0^\infty \int_0^t f(t-u)\,g(u)\,du\; e^{-st}\,dt$$

$$= \int_0^\infty \int_u^\infty f(t-u)\,g(u)\,e^{-st}\,dt\,du$$

change of variable in t integral
$v = t-u \qquad dv = dt$

$$= \int_0^\infty \int_0^\infty f(v)\,g(u)\,\underbrace{e^{-s(v+u)}}_{e^{-sv}e^{-su}}\,dv\,du$$

$$= \int_0^\infty f(v)e^{-sv}\,dv \cdot \int_0^\infty g(u)\,e^{-su}\,du = \mathcal{L}[f]\,\mathcal{L}[g].$$

Ex Solve $\frac{d^2y}{dt^2} + 4y = e^t$, $y(0)=0$, $y'(0)=0$

$$s^2\mathcal{L}[y] - \cancel{y'(0)} - \cancel{s\,y(0)} + 4\mathcal{L}[y] = \frac{1}{s-1}$$

$$(s^2+4)\mathcal{L}[y] = \frac{1}{s-1}$$

$$\mathcal{L}[y] = \frac{1}{s^2+4}\cdot\frac{1}{s-1}$$

$$\mathcal{L}[y] = \underbrace{\frac{1}{s^2+4}}_{\mathcal{L}[f]} \cdot \underbrace{\frac{1}{s-1}}_{\mathcal{L}[g]} \qquad = \frac{As+B}{s^2+4} + \frac{C}{s-1} \quad \ldots$$

$$f = \frac{1}{2}\sin 2t \qquad g = e^t$$

$$\Rightarrow \quad y = f * g = \int_0^t \frac{1}{2}\sin(2(t-u))\, e^u \, du$$

Generally, for $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$, $y(0)=0$, $y'(0)=0$

$$s^2\mathcal{L}[y] - \cancel{y'(0)} - \cancel{s\,y(0)} + p(s\mathcal{L}[y] - \cancel{y(0)}) + q\mathcal{L}[y] = \mathcal{L}[g]$$

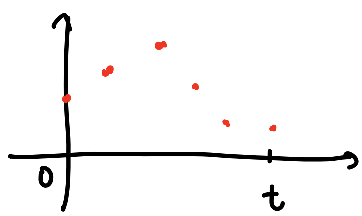
$$(s^2+ps+q)\mathcal{L}[y] = \mathcal{L}[g]$$

$$\mathcal{L}[y] = \underbrace{\frac{1}{s^2+ps+q}}_{\mathcal{L}[f]}\mathcal{L}[g]$$

$$f = \mathcal{L}^{-1}\left[\frac{1}{s^2+ps+q}\right]$$

$$y = f * g = \int_0^t f(t-u)\, g(u)\, du$$

- If one only knows some data for the external forcing g, say, some point values, this formula helps us give an approximation of $y(t)$.



Review

$\mathcal{L}[u_a(t) f(t-a)] = e^{-as} F(s)$

Ex $\dfrac{dy}{dt} = -y + 2u_3(t)e^t, \quad y(0) = 1$

$u_3(t)e^t = \underbrace{u_3(t)e^{t-3}}\cdot e^3 \qquad (a=3,\ f(t)=e^t)$

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + 2e^3 e^{-3s}\frac{1}{s-1}$$

$$s\mathcal{L}[y] - 1 = -\mathcal{L}[y] + 2e^3 e^{-3s}\frac{1}{s-1}$$

$$(s+1)\mathcal{L}[y] = 1 + 2e^3 e^{-3s}\frac{1}{s-1}$$

$$\mathcal{L}[y] = \frac{1}{s+1} + 2e^3 e^{-3s}\frac{1}{(s-1)(s+1)}$$

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s-1)$$

$s=-1 \Rightarrow 1 = -2B \qquad B = -\frac{1}{2}$

$s=1 \Rightarrow 1 = 2A \qquad A = \frac{1}{2}$

$$\mathcal{L}[y] = \frac{1}{s+1} + 2e^3e^{-3s}\cdot\frac{1}{2}\cdot\frac{1}{s-1} + 2e^3e^{-3s}\cdot\left(-\frac{1}{2}\right)\frac{1}{s+1}$$

$$= \frac{1}{s+1} + e^3e^{-3s}\frac{1}{s-1} - e^3e^{-3s}\frac{1}{s+1}$$

$a=3,\ F=\frac{1}{s-1},\ f=e^t \qquad a=3,\ F=\frac{1}{s+1},\ f=e^{-t}$

$$y = e^{-t} + e^3\cdot u_3(t)e^{t-3} - e^3\cdot u_3(t)e^{-(t-3)}$$

$\mathcal{L}[u_a(t) f(t-a)] = e^{-as} F(s)$