6.2 (continued)

Ex Solve $\frac{dy}{dt} = \begin{cases} 2y + e^{2t} & 0 \le t < 1 \\ 2y - e^{2t} & t \ge 1 \end{cases}$, $y(0) = 0$

$$= 2y + e^{2t} + u_1(t)\,(-2e^{2t})$$

$$\boxed{\begin{aligned} &\mathcal{L}[u_a(t) f(t-a)] = e^{-as} F(s) \\ &\text{if } \quad \mathcal{L}[f] = F \end{aligned}}$$

$$u_1(t)\, e^{2t} = \underbrace{u_1(t)\, e^{2(t-1)}} \cdot e^2$$

$$s\,\mathcal{L}[y] - y(0) = 2\,\mathcal{L}[y] + \frac{1}{s-2} - 2e^2 \underbrace{e^{-s} \cdot \frac{1}{s-2}}$$

$$(s-2)\,\mathcal{L}[y] = \frac{1}{s-2} - 2e^2 e^{-s} \cdot \frac{1}{s-2}$$

$$\mathcal{L}[y] = \frac{1}{(s-2)^2} - 2e^2 e^{-s} \frac{1}{(s-2)^2}$$

$$\boxed{\mathcal{L}[t e^{at}] = \frac{1}{(s-a)^2}}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] = t e^{2t}$$

$$\mathcal{L}^{-1}\left[e^{-s} \frac{1}{(s-2)^2}\right] = u_1(t)\,(t-1)\,e^{2(t-1)}$$

$$y = t e^{2t} - 2e^2 u_1(t)\,(t-1)\,e^{2(t-1)}$$

Ex Solve $\quad \dfrac{dy}{dt} = \begin{cases} -y & 0 \le t < 2 \\ -y + t & t \ge 2 \end{cases}, \quad y(0) = -1$

$$= -y + \underline{\quad 0 \quad} + u_2(t)\,\underline{\quad t \quad}$$

$$= -y + u_2(t)\,t$$

$$u_2(t)\,t = \underset{a=2}{u_2}(t)\cdot\underbrace{(t-2)}_{f(t)=t} + 2\,u_2(t)$$

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + e^{-2s}\cdot\frac{1}{s^2} + 2e^{-2s}\cdot\frac{1}{s}$$

$$s\mathcal{L}[y] + 1 = -\mathcal{L}[y] + e^{-2s}\cdot\frac{1}{s^2} + 2e^{-2s}\cdot\frac{1}{s}$$

$$(s+1)\mathcal{L}[y] = -1 + e^{-2s}\cdot\frac{1}{s^2} + 2e^{-2s}\cdot\frac{1}{s}$$

$$\mathcal{L}[y] = -\frac{1}{s+1} + e^{-2s}\frac{1}{s^2(s+1)} + 2e^{-2s}\frac{1}{s(s+1)}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

$$s = 0 \Rightarrow 1 = A$$

$$s = -1 \Rightarrow 1 = B\cdot(-1) \qquad B = -1$$

$$\frac{1}{s^2(s+1)} = \frac{C}{s^2} + \frac{D}{s} + \frac{E}{s+1}$$

$$1 = C(s+1) + Ds(s+1) + Es^2$$

$$s = 0 \Rightarrow 1 = C$$

$$s = -1 \Rightarrow 1 = E$$

$$s^2 \text{ coeff} \Rightarrow 0 = D + E \qquad D = -1$$

$$\mathcal{L}[y] = -\frac{1}{s+1} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) + 2e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= -\frac{1}{s+1} + e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{1}{s} - e^{-2s}\frac{1}{s+1}$$

$$y = -e^{-t} + u_2(t)\cdot(t-2) + u_2(t) - u_2(t)e^{-(t-2)}$$

Ex $$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = u_1(t)\,e^{3t}, \quad y(0) = -1, \quad y'(0) = 0$$

$\hookrightarrow u_1(t)\cdot e^{3(t-1)}\cdot e^3$

$$s^2\mathcal{L}[y] - y'(0) - s\,y(0) + s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] = e^3 e^{-s}\cdot\frac{1}{s-3}$$

$$s^2\mathcal{L}[y] \quad + s \quad + s\mathcal{L}[y] + 1 - 2\mathcal{L}[y] = e^3 e^{-s}\cdot\frac{1}{s-3}$$

$$\underbrace{(s^2 + s - 2)}_{(s+2)(s-1)}\mathcal{L}[y] = -s - 1 + e^3 e^{-s}\frac{1}{s-3}$$

$$\mathcal{L}[y] = \frac{-s-1}{(s+2)(s-1)} + e^3 e^{-s}\frac{1}{(s-3)(s+2)(s-1)}$$

$$\frac{-s-1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$-s-1 = A(s-1) + B(s+2)$$

$$s = 1 \Rightarrow -2 = 3B \qquad B = -\frac{2}{3}$$

$$s = -2 \Rightarrow 1 = -3A \qquad A = -\frac{1}{3}$$

$$\frac{1}{(s-3)(s+2)(s-1)} = \frac{C}{s-3} + \frac{D}{s+2} + \frac{E}{s-1}$$

$$1 = C(s+2)(s-1) + D(s-3)(s-1) + E(s-3)(s+2)$$

$$s = -2 \Rightarrow 1 = D\cdot(-5)\cdot(-3) \qquad D = \frac{1}{15}$$

$s = 1 \Rightarrow 1 = E\cdot(-2)\cdot 3$ $\qquad E = -\frac{1}{6}$

$s = 3 \Rightarrow 1 = C\cdot 5\cdot 2$ $\qquad C = \frac{1}{10}$

$$\mathcal{L}[y] = -\frac{1}{3}\cdot\frac{1}{s+2} - \frac{2}{3}\cdot\frac{1}{s-1} + e^3 e^{-s}\left(\frac{1}{10}\cdot\frac{1}{s-3} + \frac{1}{15}\cdot\frac{1}{s+2} - \frac{1}{6}\frac{1}{s-1}\right)$$

$$= -\frac{1}{3}\frac{1}{s+2} - \frac{2}{3}\cdot\frac{1}{s-1} + e^3\cdot\frac{1}{10}\cdot e^{-s}\frac{1}{s-3} + e^3\cdot\frac{1}{15}\cdot e^{-s}\frac{1}{s+2}$$

$$- e^3\cdot\frac{1}{6}\cdot e^{-s}\frac{1}{s-1}$$

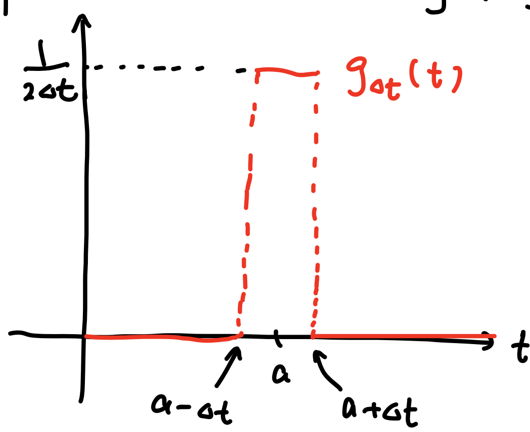
$$\Rightarrow y = -\frac{1}{3}e^{-2t} - \frac{2}{3}e^{t} + e^3\cdot\frac{1}{10}\cdot u_1(t)\, e^{3(t-1)}$$

$$+ e^3\cdot\frac{1}{15}\cdot u_1(t)\cdot e^{-2(t-1)} - e^3\cdot\frac{1}{6}\, u_1(t)\cdot e^{t-1}$$

6.4 Delta function and impulse forcing

- An impulse forcing is an external force which is very large but lasts for a very short time.

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g_{\Delta t}(t)$$



- The "limit" as $\Delta t \to 0$ of $g_{\Delta t}(t)$ is the Dirac delta function at a, denoted as $\delta_a(t)$

- $\delta_a(t) = \begin{cases} 0 & \text{if } t \neq a \\ \infty & \text{if } t = a \end{cases}$ $\quad$ w/ $\int_0^\infty \delta_a(t)\,dt = 1$

- $\delta_a(t)$ is not a function, but it can work similarly as functions.