6.3 (continued)

- $$\boxed{\begin{aligned}\mathcal{L}[t\cos\omega t] &= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}\\ \mathcal{L}[t\sin\omega t] &= \frac{2\omega s}{(s^2+\omega^2)^2}\end{aligned}}$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}$$

$$\mathcal{L}[te^{i\omega t}] = \frac{1}{(s-i\omega)^2} = \frac{(s+i\omega)^2}{(s-i\omega)^2(s+i\omega)^2}$$

$$= \frac{s^2-\omega^2+2si\omega}{(s^2+\omega^2)^2}$$

Ex Solve $\frac{d^2y}{dt^2} + 9y = \cos 3t$, $y(0)=1$, $y'(0)=0$

Take $\mathcal{L}$,

$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + 9\mathcal{L}[y] = \frac{s}{s^2+9}$$

$$s^2\mathcal{L}[y] - s + 9\mathcal{L}[y] = \frac{s}{s^2+9}$$

$$(s^2+9)\mathcal{L}[y] = \frac{s}{s^2+9} + s$$

$\mathcal{L}[t\sin 3t] = \frac{6s}{(s^2+9)^2}$

$$\mathcal{L}[y] = \frac{s}{(s^2+9)^2} + \frac{s}{s^2+9}$$

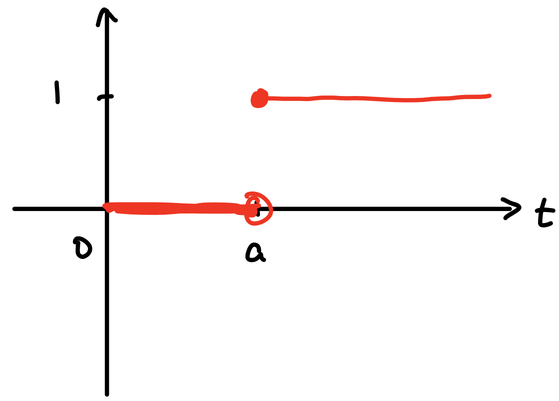
$$y = \frac{1}{6}t\sin 3t + \cos 3t$$

6.2 Discontinuous functions

- The <u>Heaviside function</u>

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

where $a \geq 0$ is a parameter



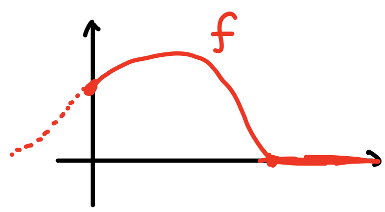
$$\mathcal{L}[u_a] = \int_0^\infty u_a(t)\, e^{-st}\, dt = \int_a^\infty e^{-st}\, dt = \frac{1}{-s} e^{-st}\Big|_a^\infty$$

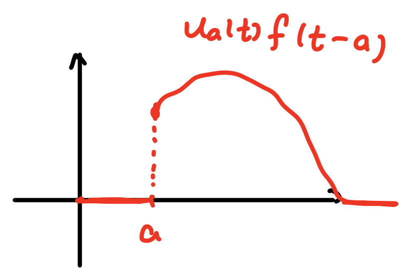
for $s > 0$

$$= \frac{1}{-s}\left(0 - e^{-as}\right) = \frac{1}{s} e^{-as}$$

$$\boxed{\mathcal{L}[u_a] = \frac{1}{s} e^{-as}}$$

- If $\mathcal{L}[f] = F(s)$ then





$$\mathcal{L}[u_a(t) f(t-a)]$$

$$= \int_0^\infty u_a(t) f(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt$$

$v = t - a$

$dv = dt$

$$= \int_0^\infty f(v)\, e^{-s(v+a)}\, dv$$

$$= \int_0^\infty f(v)\, e^{-sv} e^{-sa}\, dv = e^{-sa} \int_0^\infty f(v)\, e^{-sv}\, dv = e^{-sa} F(s)$$

$$\boxed{\mathcal{L}[f] = F(s) \Rightarrow \mathcal{L}[u_a(t) f(t-a)] = e^{-as} F(s)}$$

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s+3}\right] = u_2(t)e^{-3(t-2)}$$

$a=2$

$F(s)=\frac{1}{s+3}$

$f(t)=e^{-3t}$

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s^2+1}\right] = u_3(t)\sin(t-3)$$

$a=3$

$F(s)=\frac{1}{s^2+1}$

$f(t)=\sin t$

$\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}$

Ex Solve $\frac{dy}{dt} = 2y - u_3(t)$, $y(0) = -2$

Take $\mathcal{L}$,

$$s\mathcal{L}[y] - y(0) = 2\mathcal{L}[y] - \frac{1}{s}e^{-3s}$$

$$s\mathcal{L}[y] + 2 = 2\mathcal{L}[y] - \frac{1}{s}e^{-3s}$$

$$(s-2)\mathcal{L}[y] = -\frac{1}{s}e^{-3s} - 2$$

$$\mathcal{L}[y] = -\frac{1}{s(s-2)}e^{-3s} - \frac{2}{s-2}$$

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$1 = A(s-2) + Bs$$

$s=2 \Rightarrow 1 = 2B \qquad B=\frac{1}{2}$

$s=0 \Rightarrow 1 = -2A \qquad A=-\frac{1}{2}$

$$\frac{1}{s(s-2)}e^{-3s} = -\frac{1}{2}\cdot\underbrace{\frac{1}{s}}_{\substack{F(s)=\frac{1}{s}\\ f(t)=1}}\underbrace{e^{-3s}}_{a=3} + \frac{1}{2}\cdot\underbrace{\frac{1}{s-2}}_{\substack{F(s)=\frac{1}{s-2}\\ f(t)=e^{2t}}}\underbrace{e^{-3s}}_{a=3}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s-2)}e^{-3s}\right] = -\frac{1}{2}\cdot u_3(t) + \frac{1}{2}\cdot u_3(t)\,e^{2(t-3)}$$

$$y = \frac{1}{2}u_3(t) - \frac{1}{2}u_3(t)\,e^{2(t-3)} - 2e^{2t}$$

- $$\boxed{\mathcal{L}[f] = F(s) \Rightarrow \mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)}$$

$$u_3(t)e^{-2t}$$

$$= u_3(t)\,e^{\underbrace{-2(t-3)}_{-2t+6}}\cdot e^{-6}$$

$$\mathcal{L}[u_3(t)e^{-2t}]$$

$$= \mathcal{L}[u_3(t)e^{-2(t-3)}e^{-6}]$$

$$= e^{-6}\cdot\mathcal{L}[\underbrace{u_3(t)}_{a=3}\underbrace{e^{-2(t-3)}}_{\substack{f(t)=e^{-2t}\\ F(s)=\frac{1}{s+2}}}]$$

$$= e^{-6}\cdot e^{-3s}\frac{1}{s+2}$$

<u>Ex</u> Solve $\frac{dy}{dt} = -y + u_2(t)e^{3t}$, $\quad y(0)=0$

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + e^{6}e^{-2s}\frac{1}{s-3}$$

$$s\mathcal{L}[y] = -\mathcal{L}[y] + e^6 e^{-2s}\frac{1}{s-3}$$

$$(s+1)\mathcal{L}[y] = e^6 e^{-2s}\frac{1}{s-3}$$

$$\mathcal{L}[y] = e^6 e^{-2s}\frac{1}{(s-3)(s+1)}$$

$$\mathcal{L}[u_2(t)e^{3t}]$$

$$= \mathcal{L}[u_2(t)e^{3(t-2)}\cdot e^6]$$

$$= e^6 e^{-2s}\frac{1}{s-3}$$

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s-3)$$

$$s=-1 \Rightarrow 1 = -4B \qquad B = -\frac{1}{4}$$

$$s=3 \Rightarrow 1 = 4A \qquad A = \frac{1}{4}$$

$$e^6 e^{-2s}\frac{1}{(s-3)(s+1)} = \frac{1}{4}e^6 e^{-2s}\cdot\frac{1}{s-3} - \frac{1}{4}e^6 e^{-2s}\cdot\frac{1}{s+1}$$

$$y = \mathcal{L}^{-1}\left[e^6 e^{-2s}\frac{1}{(s-3)(s+1)}\right] = \frac{1}{4}e^6 u_2(t)\cdot e^{3(t-2)} - \frac{1}{4}e^6 u_2(t)\cdot e^{-(t-2)}$$

__Ex__ Solve $\frac{dy}{dt} = \begin{cases} 2y + e^{2t} & 0 \le t < 1 \\ 2y - e^{2t} & t \ge 1 \end{cases}$, $y(0)=0$

$$= 2y + \underline{e^{2t}} + u_1(t)\,\underline{(-2e^{2t})}$$