Midterm 2 : average = 82

Final exam : 3:30 ~ 5:30pm May 7th

same as lecture room

accummulative, weighted more on Chapter 6

8 problems

6.3 second order equations

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$

Recall : $\mathcal{L}[\frac{dy}{dt}] = s\mathcal{L}[y] - y(0)$

$\mathcal{L}[\frac{d^2y}{dt}] = s^2\mathcal{L}[y] - y'(0) - s\,y(0)$

Ex Solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$, $y(0) = 3$, $y'(0) = -2$

Take $\mathcal{L}$,

$\mathcal{L}[e^{at}] = \frac{1}{s-a}$

$$s^2\mathcal{L}[y] - y'(0) - s\,y(0) + 3(s\mathcal{L}[y] - y(0)) - 4\mathcal{L}[y] = \frac{1}{s-2}$$

$$s^2\mathcal{L}[y] - (-2) - 3s + 3(s\mathcal{L}[y] - 3) - 4\mathcal{L}[y] = \frac{1}{s-2}$$

$$(s^2 + 3s - 4)\mathcal{L}[y] = \frac{1}{s-2} - 2 + 3s + 9$$

$$(s+4)(s-1)\mathcal{L}[y] = \frac{1}{s-2} + 3s + 7$$

$$\mathcal{L}[y] = \frac{1}{(s-2)(s+4)(s-1)} + \frac{3s+7}{(s+4)(s-1)}$$

$$\frac{3s+7}{(s+4)(s-1)} = \frac{A}{s+4} + \frac{B}{s-1}$$

$$3s+7 = A(s-1) + B(s+4)$$

$$s=1 \Rightarrow 10 = 5B \Rightarrow B=2$$

$$s=-4 \Rightarrow -5 = -5A \Rightarrow A=1$$

$$\frac{1}{(s-2)(s+4)(s-1)} = \frac{C}{s-2} + \frac{D}{s+4} + \frac{E}{s-1}$$

$$1 = C(s+4)(s-1) + D(s-2)(s-1) + E(s-2)(s+4)$$

$$s=2 \Rightarrow 1 = C\cdot 6\cdot 1 \Rightarrow C = \frac{1}{6}$$

$$s=-4 \Rightarrow 1 = D\cdot(-6)\cdot(-5) \Rightarrow D = \frac{1}{30}$$

$$s=1 \Rightarrow 1 = E\cdot(-1)\cdot 5 \Rightarrow E = -\frac{1}{5}$$

$$\mathcal{L}[y] = \frac{1}{s+4} + 2\cdot\frac{1}{s-1} + \frac{1}{6}\cdot\frac{1}{s-2} + \frac{1}{30}\cdot\frac{1}{s+4} - \frac{1}{5}\cdot\frac{1}{s-1}$$

$$= \frac{31}{30}\cdot\frac{1}{s+4} + \frac{9}{5}\cdot\frac{1}{s-1} + \frac{1}{6}\cdot\frac{1}{s-2}$$

$$y = \frac{31}{30}e^{-4t} + \frac{9}{5}e^{t} + \frac{1}{6}e^{2t}$$

• $\mathcal{L}$ of cos and sin

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2}$$

Take real part: $\boxed{\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}}$

Take imaginary part: $\boxed{\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}}$

Ex Solve $\frac{d^2y}{dt^2} + 4y = 0$, $y(0)=0$, $y'(0)=1$

Take $\mathcal{L}$,

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + 4\mathcal{L}[y] = 0$$

$$s^2\mathcal{L}[y] - 1 + 4\mathcal{L}[y] = 0$$

$$(s^2+4)\mathcal{L}[y] = 1$$

$$\mathcal{L}[y] = \frac{1}{s^2+4}$$

Take $\omega = 2$

$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$

$$y = \frac{1}{2}\sin 2t$$

• $\boxed{\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}}$

$\boxed{\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}}$

$$\mathcal{L}[e^{(a+i\omega)t}] = \frac{1}{s-(a+i\omega)} = \frac{1}{(s-a)-i\omega} = \frac{(s-a)+i\omega}{(s-a)^2+\omega^2}$$

Ex Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 1$, $y(0)=1$, $y'(0)=0$

Take $\mathcal{L}$,

$$s^2\mathcal{L}[y] - y'(0) - s\,y(0) + 2(s\mathcal{L}[y] - y(0)) + 10\mathcal{L}[y] = \frac{1}{s}$$

$$s^2\mathcal{L}[y] - s + 2(s\mathcal{L}[y] - 1) + 10\mathcal{L}[y] = \frac{1}{s}$$

$$(s^2+2s+10)\mathcal{L}[y] - s - 2 = \frac{1}{s}$$

$$(s^2+2s+10)\mathcal{L}[y] = \frac{1}{s} + s + 2$$

$$\mathcal{L}[y] = \frac{1}{s(s^2+2s+10)} + \frac{s+2}{s^2+2s+10}$$

"$(s-a)^2+\omega^2$"

$s^2+2s+10$

$= (s+1)^2+9$

$a=-1,\ \omega=3$

$$\mathcal{L}[e^{-t}\cos 3t] = \frac{s+1}{(s+1)^2+9}$$

$$\mathcal{L}[e^{-t}\sin 3t] = \frac{3}{(s+1)^2+9}$$

$$\frac{s+2}{s^2+2s+10} = \frac{(s+1)+1}{(s+1)^2+9}$$

$$= \frac{(s+1) + \frac{1}{3}\cdot 3}{(s+1)^2+9}$$

$$\mathcal{L}^{-1}\left[\frac{s+2}{s^2+2s+10}\right] = e^{-t}\cos 3t + \frac{1}{3}e^{-t}\sin 3t$$

$$\frac{1}{s(s^2+2s+10)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+10}$$

$$1 = A(s^2+2s+10) + (Bs+C)s$$

$s=0 \Rightarrow 1 = 10A \Rightarrow A = \frac{1}{10}$

s^2 coeff $\Rightarrow 0 = A+B \Rightarrow B = -\frac{1}{10}$

s coeff $\Rightarrow 0 = 2A + C \Rightarrow C = -\frac{1}{5}$

$$\frac{1}{s(s^2+2s+10)} = \frac{1}{10}\cdot\frac{1}{s} + \frac{-\frac{1}{10}s - \frac{1}{5}}{(s+1)^2+9}$$

$$= \frac{1}{10}\cdot\frac{1}{s} + \frac{-\frac{1}{10}(s+1) - \frac{1}{10}}{(s+1)^2+9}$$

$$= \frac{1}{10} \cdot \frac{1}{s} + \frac{-\frac{1}{10}(s+1) - \frac{1}{30} \cdot 3}{(s+1)^2 + 9}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+2s+10)}\right] = \frac{1}{10} - \frac{1}{10} e^{-t} \cos 3t - \frac{1}{30} e^{-t} \sin 3t$$

$$\Rightarrow \quad y = \frac{1}{10} - \frac{1}{10} e^{-t} \cos 3t - \frac{1}{30} e^{-t} \sin 3t$$

$$+ e^{-t} \cos 3t + \frac{1}{3} e^{-t} \sin 3t$$

$$= \frac{1}{10} + \frac{9}{10} e^{-t} \cos 3t + \frac{3}{10} e^{-t} \sin 3t$$