6.1 Laplace transform

Def For a function $y(t)$ defined on $[0,\infty)$, its Laplace transform is a function $Y(s)$ given by

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

(for all real numbers s such that the integral converges).

Notation: $Y = \mathcal{L}[y]$

- Y is in a new variable s, not the original variable t.

Ex Calculate $\mathcal{L}[e^{3t}]$

(assuming $s>3$)

$$\mathcal{L}[e^{3t}] = \int_0^\infty e^{3t} e^{-st}\, dt = \int_0^\infty e^{(3-s)t}\, dt$$

$$= \frac{1}{3-s} e^{(3-s)t}\Big|_{t=0}^{\infty} = \frac{1}{3-s}(0-1) = \frac{1}{s-3}$$

- Generally, $\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad (\text{for } s>a)}$

In particular, $\mathcal{L}[1] = \frac{1}{s}$ (for $s>0$).

Properties

- Linear property: $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$

$\mathcal{L}[cf] = c\,\mathcal{L}[f]$ (where c is a scalar).

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

$A(c\vec{x}) = cA\vec{x}$

Proof of $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$

$$\mathcal{L}[f+g] = \int_0^\infty (f(t)+g(t))\, e^{-st}\,dt$$

$$= \int_0^\infty f(t) e^{-st}\,dt + \int_0^\infty g(t) e^{-st}\,dt$$

$$= \mathcal{L}[f] + \mathcal{L}[g].$$

Ex $\mathcal{L}[2e^{3t} - e^{-4t}] = 2\mathcal{L}[e^{3t}] - \mathcal{L}[e^{-4t}]$

$$= 2\cdot\frac{1}{s-3} - \frac{1}{s+4}.$$

- Laplace transform of derivatives:

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty y'(t)\, e^{-st}\,dt$$

$u = e^{-st}$ $\quad v = y(t)$

$du = -se^{-st}dt$ $\quad dv = y'(t)dt$

$$= e^{-st} y(t)\Big|_0^\infty - \int_0^\infty y(t)(-se^{-st})\,dt$$

$$= (0 - y(0)) + s\int_0^\infty y(t) e^{-st}\,dt$$

$$= s\,\mathcal{L}[y] - y(0)$$

"$\mathcal{L}$ converts derivatives into algebraic operations"

Ex Use $\mathcal{L}$ to solve

$$\frac{dy}{dt} = -2y + e^{3t}, \quad y(0) = -1$$

Take $\mathcal{L}$,

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -2\mathcal{L}[y] + \mathcal{L}[e^{3t}]$$

$$s\mathcal{L}[y] - y(0) = -2\mathcal{L}[y] + \frac{1}{s-3}$$

$$s\mathcal{L}[y] + 1 = -2\mathcal{L}[y] + \frac{1}{s-3}$$

$$(s+2)\mathcal{L}[y] = -1 + \frac{1}{s-3}$$

$$\mathcal{L}[y] = -\frac{1}{s+2} + \frac{1}{(s-3)(s+2)}$$

partial fraction: $\dfrac{1}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2}$

$$1 = A(s+2) + B(s-3)$$

Take $s = -2$: $1 = -5B \Rightarrow B = -\frac{1}{5}$

Take $s = 3$: $1 = 5A \Rightarrow A = \frac{1}{5}$

$$\Rightarrow \mathcal{L}[y] = -\frac{1}{s+2} + \frac{1}{5}\cdot\frac{1}{s-3} - \frac{1}{5}\cdot\frac{1}{s+2}$$

$$= -\frac{6}{5}\cdot\frac{1}{s+2} + \frac{1}{5}\cdot\frac{1}{s-3}$$

$$\Rightarrow y = -\frac{6}{5}\cdot e^{-2t} + \frac{1}{5}\cdot e^{3t}.$$

inverse Laplace transform

$$\mathcal{L}^{-1}\left[-\frac{1}{s+2} + \frac{1}{(s-3)(s+2)}\right] = -\frac{6}{5}e^{-2t} + \frac{1}{5}e^{3t}$$

Ex Use $\mathcal{L}$ to solve

$$\frac{dy}{dt} = y + 2 - 3e^{-2t}, \quad y(0) = 1$$

Take $\mathcal{L}$,

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[y] + 2\mathcal{L}[1] - 3\mathcal{L}[e^{-2t}]$$

$$s\mathcal{L}[y] - y(0) = \mathcal{L}[y] + 2\cdot\frac{1}{s} - 3\cdot\frac{1}{s+2}$$

$$s\mathcal{L}[y] - 1 = \mathcal{L}[y] + 2\cdot\frac{1}{s} - 3\cdot\frac{1}{s+2}$$

$$(s-1)\mathcal{L}[y] = 1 + 2\cdot\frac{1}{s} - 3\cdot\frac{1}{s+2}$$

$$\mathcal{L}[y] = \frac{1}{s-1} + 2\cdot\frac{1}{s(s-1)} - 3\cdot\frac{1}{(s+2)(s-1)}$$

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{(s-1)}$$

$$1 = A(s-1) + Bs$$

Take $s = 1$: $\quad 1 = B$

Take $s = 0$: $\quad 1 = A\cdot(-1) \Rightarrow A = -1$

$$\frac{1}{(s+2)(s-1)} = \frac{C}{s+2} + \frac{D}{s-1}$$

$$1 = C(s-1) + D(s+2)$$

Take $s = 1$: $\quad 1 = 3D \Rightarrow D = \frac{1}{3}$

Take $s = -2$: $\quad 1 = -3C \Rightarrow C = -\frac{1}{3}$

$$\mathcal{L}[y] = \frac{1}{s-1} + 2\cdot\left(-\frac{1}{s} + \frac{1}{s-1}\right) - 3\left(-\frac{1}{3}\cdot\frac{1}{s+2} + \frac{1}{3}\cdot\frac{1}{s-1}\right)$$

$$= \frac{1}{s-1} - 2\cdot\frac{1}{s} + 2\cdot\frac{1}{s-1} + \frac{1}{s+2} - \frac{1}{s-1}$$

$$= 2\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s} + \frac{1}{s+2}$$

$$\Rightarrow \quad y = 2\cdot e^{t} - 2 + e^{-2t}$$

- $$\mathcal{L}[t e^{at}] = \int_0^\infty t\, e^{at} e^{-st}\, dt = \int_0^\infty t\, e^{(a-s)t}\, dt$$

$$= t\cdot\frac{1}{a-s} e^{(a-s)t}\Big|_{t=0}^{\infty} - \int_0^\infty \frac{1}{a-s} e^{(a-s)t}\, dt$$

$$\boxed{\begin{array}{ll} u = t & v = \frac{1}{a-s} e^{(a-s)t} \\ du = dt & dv = e^{(a-s)t}\, dt \end{array}}$$

$$= (0-0) - \frac{1}{a-s}\cdot\frac{1}{s-a}$$

$$= \frac{1}{(s-a)^2}$$

Ex Use $\mathcal{L}$ to solve

$$\frac{dy}{dt} = 2y + 3e^{2t}, \quad y(0) = -2$$

Take $\mathcal{L}$,

$$\mathcal{L}\left[\frac{dy}{dt}\right] = 2\mathcal{L}[y] + 3\cdot\frac{1}{s-2}$$

$$s\mathcal{L}[y] - y(0) = 2\mathcal{L}[y] + 3\cdot\frac{1}{s-2}$$

$$s\mathcal{L}[y] + 2 = 2\mathcal{L}[y] + 3\cdot\frac{1}{s-2}$$

$$(s-2)\mathcal{L}[y] = -2 + 3\cdot\frac{1}{s-2}$$

$$\mathcal{L}[y] = -2\cdot\frac{1}{s-2} + 3\cdot\frac{1}{(s-2)^2}$$

$$y = -2\cdot e^{2t} + 3te^{2t}$$

- $\mathcal{L}[\frac{d^2y}{dt^2}] = \mathcal{L}[(y')'] = s\mathcal{L}[y'] - y'(0)$

$$= s(s\mathcal{L}[y] - y(0)) - y'(0)$$

$$= s^2\mathcal{L}[y] - sy(0) - y'(0)$$

Summary: Laplace transform can be used to solve linear DE (possibly higher order, inhomogeneous) initial value problems