4.3 Undamped forcing and resonance

Ex Find general sol'n :

$$\frac{d^2y}{dt^2} + 9y = \cos(wt) \quad \text{where } w>0 \text{ is a constant} \quad \cdots\cdots (1)$$

Consider

$$\frac{d^2y}{dt^2} + 9y = e^{iwt} \quad \cdots\cdots (2)$$

$$s^2 + 9 = 0$$

$$s^2 = -9$$

$$s = \pm 3i$$

- If $w \neq 3$, then

Try $y(t) = Ce^{iwt}$ for (2)

$$\frac{dy}{dt} = iwCe^{iwt}$$

$$\frac{d^2y}{dt^2} = -w^2Ce^{iwt}$$

$$-w^2Ce^{iwt} + 9Ce^{iwt} = e^{iwt}$$

$$-w^2C + 9C = 1$$

$$C(9-w^2) = 1$$

$$C = \frac{1}{9-w^2}$$

$\Rightarrow$ particular sol'n to (2) :

$$y(t) = \frac{1}{9-\omega^2} \cdot e^{i\omega t} = \frac{1}{9-\omega^2}\left(\cos(\omega t) + i\sin(\omega t)\right)$$

$$= \frac{1}{9-\omega^2}\cos(\omega t) + i\,\frac{1}{9-\omega^2}\sin(\omega t)$$

Take real part,

$\Rightarrow$ particular sol'n to (1):

$$y(t) = \frac{1}{9-\omega^2}\cos(\omega t)$$

$\Rightarrow$ general sol'n to (1):

$$\underline{y(t) = \frac{1}{9-\omega^2}\cos(\omega t) + C_1\cos(3t) + C_2\sin(3t)}$$

- If $\omega = 3$, then

try $y(t) = C\,t\,e^{i\cdot 3t}$ for (2)

$$\frac{dy}{dt} = C\left(e^{3it} + 3i\,t\,e^{3it}\right)$$

$$\frac{d^2y}{dt^2} = C\left(3i\,e^{3it} + 3i\,e^{3it} - 9t\,e^{3it}\right)$$

$$= C\left(6i\,e^{3it} - 9t\,e^{3it}\right)$$

$$C\left(6i\,e^{3it} - 9t\,e^{3it}\right) + 9C\,t\,e^{3it} = e^{3it}$$

$$C(6i - 9t) + 9Ct = 1$$

$$C(6i \cancel{-9t} \cancel{+9t}) = 1$$

$$C = \frac{1}{6i} = -\frac{1}{6}i$$

$\Rightarrow$ particular sol'n to (2):

$$y(t) = -\frac{1}{6} i t e^{3it}$$
$$= -\frac{1}{6} i t \left(\cos(3t) + i \sin(3t) \right)$$
$$= -\frac{1}{6} i t \cos(3t) + \frac{1}{6} t \sin(3t)$$

Take real part

$\Rightarrow$ particular sol'n to (1):

$$y(t) = \frac{1}{6} t \sin(3t)$$

$\Rightarrow$ general sol'n to (1):

$$\underline{y(t) = \frac{1}{6} t \sin(3t) + C_1 \cos(3t) + C_2 \sin(3t)}$$

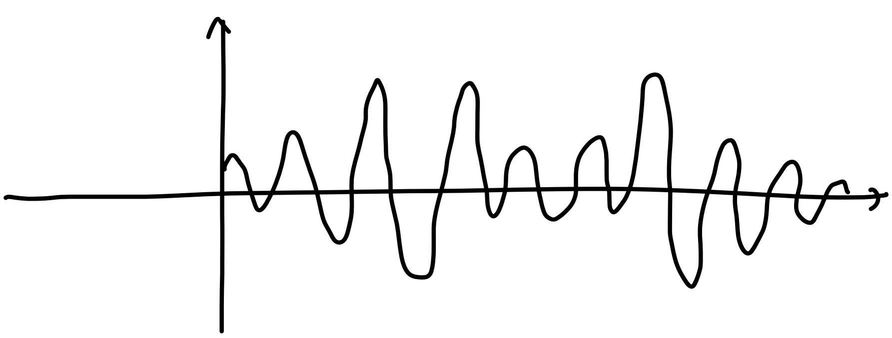
Qualitative behavior

- When $\omega \neq 3$ (that is, ω not equal to natural frequency),

$$y(t) = \underbrace{\frac{1}{9-\omega^2} \cos(\omega t)}_{\text{oscillation w/ external force frequency}} + \underbrace{C_1 \cos(3t) + C_2 \sin(3t)}_{\text{oscillation w/ natural frequency}}$$

$y(t)$ is a linear superposition of oscillations w/ different frequency

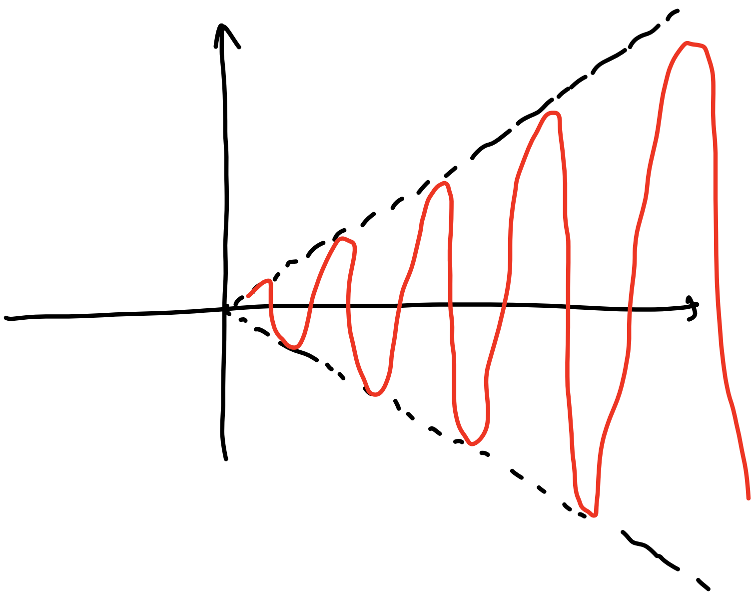
bounded in all time w/ oscillation



- When $\omega = 3$ (ω equal to natural frequency)

$$y(t) = \underbrace{\frac{1}{6} t \sin(3t)}_{\substack{\text{oscillation w/} \\ \text{larger and larger amplitude} \\ \text{as } t \to \infty .}} + C_1 \cos(3t) + C_2 \sin(3t)$$

As $t \to \infty$, $y(t)$ oscillates w/ larger and larger amplitude (which grows linearly in t), unbounded.



"resonance"

- When ω is close to 3 (natural frequency) but not equal.

Bounded oscillation w/ a large amplitude.

4.4 Amplitude and phase of steady states (for forced damped harmonic oscillator)

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t) \quad p, q > 0, \ \omega > 0 \quad \text{- - - - (3)}$$

Find a particular sol'n :

Consider $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{i\omega t} \quad \text{- - - - (4)}$$

(know $i\omega$ is not eigenvalue)

Try $y(t) = C e^{iwt}$

$$\frac{dy}{dt} = iw C e^{iwt}$$

$$\frac{d^2y}{dt^2} = -w^2 C e^{iwt}$$

$$-w^2 C e^{iwt} + piwCe^{iwt} + qCe^{iwt} = e^{iwt}$$

$$-w^2 C + piwC + qC = 1$$

$$C(-w^2 + piw + q) = 1$$

$$C = \frac{1}{(q-w^2) + ipw} = \frac{(q-w^2) - ipw}{(q-w^2)^2 + (pw)^2}$$

$\Rightarrow$ particular sol'n to (4):

$$y(t) = \frac{(q-w^2) - ipw}{(q-w^2)^2 + (pw)^2} e^{iwt}$$

$$= \frac{1}{(q-w^2)^2 + (pw)^2} \cdot ((q-w^2) - ipw)(\cos(wt) + i\sin(wt))$$

$$= \frac{1}{(q-w^2)^2 + (pw)^2} \left((q-w^2)\cos(wt) + pw\sin(wt) + i(\cdots) \right)$$

Take real part,

$\Rightarrow$ particular sol'n to (3):

$$y(t) = \frac{1}{(q-w^2)^2 + (pw)^2} \left((q-w^2)\cos(wt) + pw\sin(wt) \right)$$

Amplitude of $a\cos wt + b\sin wt$ is $A = \sqrt{a^2+b^2}$

Amplitude $A = \sqrt{\left(\frac{q-\omega^2}{(q-\omega^2)^2+(p\omega)^2}\right)^2 + \left(\frac{p\omega}{(q-\omega^2)^2+(p\omega)^2}\right)^2}$

$$= \frac{1}{(q-\omega^2)^2+(p\omega)^2} \cdot \sqrt{(q-\omega^2)^2+(p\omega)^2}$$

$$= \frac{1}{\sqrt{(q-\omega^2)^2+(p\omega)^2}}$$

When p, q fixed, A is maximized when

$(q-\omega^2)^2+(p\omega)^2$ is minimized

$$(q-\omega^2)^2+(p\omega)^2 = q^2 - 2q\omega^2 + \omega^4 + p^2\omega^2$$

$$= \omega^4 - (2q-p^2)\omega^2 + q^2$$

$$= \left(\omega^2 - \left(q-\frac{p^2}{2}\right)\right)^2 + q^2 - \left(q-\frac{p^2}{2}\right)^2$$

$$\begin{cases} \text{If } q-\frac{p^2}{2} > 0 \text{ then } A \text{ is maximized when } \omega = \sqrt{q-\frac{p^2}{2}} \\ \text{If } q-\frac{p^2}{2} \leq 0 \text{ then } A \text{ is maximized when } \omega \to 0 \text{ (not achieved)} \end{cases}$$

- If p is small, then A is almost maximized at $\omega = \sqrt{q}$ (natural frequency of undamped harmonic oscillator)

and $A|_{\omega=\sqrt{q}}$ is large $\Rightarrow$ almost resonance.