4.2 Sinusoidal forcing

Ex Find a particular sol'n to

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \cos t \quad \cdots\cdots (1)$$

$$e^{it} = \cos t + i\sin t$$

Consider

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{it} \quad \cdots\cdots (2)$$

Strategy: first find a particular sol'n (2), then take real part to get a particular sol'n to (1).

$$s^2 + 5s + 6 = 0$$

$$(s+3)(s+2) = 0$$

$$s_1 = -3, \ s_2 = -2$$

Try $y(t) = Ce^{it}$ for (2)

$$\frac{dy}{dt} = Cie^{it}$$

$$\frac{d^2y}{dt^2} = -Ce^{it}$$

$$-Ce^{it} + 5Cie^{it} + 6Ce^{it} = e^{it}$$

$$-C + 5Ci + 6C = 1$$

$$C(-1 + 5i + 6) = 1$$

$$C(5 + 5i) = 1 \qquad C = \frac{1}{5+5i}$$

$\Rightarrow$ particular sol'n of (2) : $y(t) = \frac{1}{5+5i} e^{it}$

Take real part :

$$\frac{1}{5+5i} e^{it}$$

$$= \frac{5-5i}{(5+5i)(5-5i)} e^{it}$$

$$= \frac{5-5i}{50} e^{it}$$

$$= \frac{1-i}{10} (\cos t + i \sin t)$$

$$= \frac{1}{10} \left(\cos t + i \sin t - i \cos t + \sin t \right)$$

$$= \frac{1}{10} (\cos t + \sin t) + i \cdot \frac{1}{10} (\sin t - \cos t)$$

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

$\Rightarrow$ particular sol'n to (1) : $y(t) = \frac{1}{10}(\cos t + \sin t)$.

Generally, to find a particular sol'n to

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t)$$

(or $\sin(\omega t)$)

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

First find a particular sol'n to

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{i\omega t}$$

Then take real (or imaginary) part.

Ex Find general sol'n :

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = \sin 2t \quad \cdots\cdots (3)$$

Consider

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = e^{2it} \quad \cdots\cdots (4)$$

$$s^2 + 2s + 10 = 0$$

$$s = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 10}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

Try $y(t) = Ce^{2it}$ for (4)

$$\frac{dy}{dt} = 2iCe^{2it}$$

$$\frac{d^2y}{dt^2} = -4Ce^{2it}$$

$$-4Ce^{2it} + 2\cdot 2iCe^{2it} + 10Ce^{2it} = e^{2it}$$

$$-4C + 4iC + 10C = 1$$

$$4iC + 6C = 1$$

$$C(6+4i) = 1$$

$$C = \frac{1}{6+4i} = \frac{6-4i}{(6+4i)(6-4i)}$$

$$= \frac{6-4i}{52} = \frac{3-2i}{26}$$

$\Rightarrow$ particular sol'n to (4) :

$$y(t) = \frac{3-2i}{26}e^{2it} = \frac{3-2i}{26}(\cos 2t + i\sin 2t)$$

$$= \frac{1}{26}\left(3\cos 2t + 3i\sin 2t - 2i\cos 2t + 2\sin 2t\right)$$

$$= \frac{1}{26}\left(3\cos 2t + 2\sin 2t\right) + i\cdot\frac{1}{26}\left(3\sin 2t - 2\cos 2t\right)$$

Take imaginary part

$\Rightarrow$ particular sol'n to (3):

$$y(t) = \frac{1}{26}\left(3\sin 2t - 2\cos 2t\right)$$

$\Rightarrow$ general sol'n to (3):

$$y(t) = \frac{1}{26}\left(3\sin 2t - 2\cos 2t\right) + C_1 e^{-t}\cos 3t + C_2 e^{-t}\sin 3t.$$

Qualitative behavior of sol'ns to

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t) \quad \text{w/} \quad \omega > 0$$

(or $\sin(\omega t)$) $\leadsto e^{i\omega t}$

when $p > 0$, $q > 0$.

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t) = a\cos(\omega t) + b\sin(\omega t)$

and $y_h(t) = \begin{cases} C_1 e^{s_1 t} + C_2 e^{s_2 t} \\ C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t \\ C_1 e^{st} + C_2 t e^{st} \end{cases}$

$p > 0$, $q > 0$ implies that one of the following holds

$\begin{cases} s_1, s_2 < 0 \\ s = \alpha \pm \beta i, \ \alpha < 0 \\ s < 0 \end{cases}$

$s^2 + ps + q = 0$

$s_1 + s_2 = -p$

$s_1 s_2 = q$

As $t \to \infty$, $y_p(t)$ oscillates while $y_h(t) \longrightarrow 0$ (exponentially).

$$\Rightarrow \quad y(t) \approx y_p(t) = a\cos(\omega t) + b\sin(\omega t) \quad \text{for large } t.$$

$$= A\cos(\omega t + \phi)$$

amplitude $\qquad\qquad$ phase shift.

$A = \sqrt{a^2+b^2}$

<u>Ex</u> Find a particular sol'n

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}\cos 2t \quad \cdots\cdots (5)$$

Consider

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{(-1+2i)t} \quad \cdots\cdots (6)$$

$$s^2 + 4s + 4 = 0$$

$$(s+2)^2 = 0$$

$$s = -2 \text{ (repeated)}$$

Try $\quad y(t) = Ce^{(-1+2i)t}$

$$\frac{dy}{dt} = C(-1+2i)e^{(-1+2i)t}$$

$$\frac{d^2y}{dt^2} = C(-1+2i)^2 e^{(-1+2i)t}$$

$$C(-1+2i)^2 e^{(-1+2i)t} + 4C(-1+2i)e^{(-1+2i)t} + 4Ce^{(-1+2i)t} = e^{(-1+2i)t}$$

$$C(-1+2i)^2 + 4C(-1+2i) + 4C = 1$$

$$C\left((-1+2i)^2 + 4(-1+2i) + 4\right) = 1$$

$$C\left(1-4-4i \cancel{-4} + 8i \cancel{+4}\right) = 1$$

$$C(-3+4i) = 1$$

$$C = \frac{1}{-3+4i} = \frac{-3-4i}{(-3+4i)(-3-4i)} = \frac{-3-4i}{25}$$

$\Rightarrow$ particular sol'n to (6) :

$$y(t) = \frac{-3-4i}{25} e^{(-1+2i)t}$$

$$= \frac{1}{25} e^{-t} (-3-4i)(\cos 2t + i \sin 2t)$$

$$= \frac{1}{25} e^{-t} (-3\cos 2t - 3i \sin 2t - 4i \cos 2t + 4 \sin 2t)$$

Take real part

$\Rightarrow$ particular sol'n to (5) :

$$y(t) = \frac{1}{25} e^{-t} (-3\cos 2t + 4 \sin 2t) .$$