4.1 (continued)

Recall: $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-6t}$$

has a particular sol'n $y_p(t) = \frac{1}{15} e^{-6t}$

Ex Find general sol'n to $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-6t}$

Need general sol'n to homogeneous eq.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0$$

char. eq. $$s^2 + 4s + 3 = 0$$

$$(s+1)(s+3) = 0$$

$$s_1 = -1, \quad s_2 = -3$$

$\Rightarrow$ General sol'n to homo. eq.:

$$y_h(t) = C_1 e^{-t} + C_2 e^{-3t}$$

$\Rightarrow$ General sol'n to inhomo. eq.:

$$y(t) = y_p(t) + y_h(t)$$

$$= \frac{1}{15} e^{-6t} + C_1 e^{-t} + C_2 e^{-3t}$$

Ex Find a particular sol'n to

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2e^{-3t}$$

Try $y(t) = Ce^{-3t}$

$$\frac{dy}{dt} = -3Ce^{-3t}$$

$$\frac{d^2y}{dt^2} = 9Ce^{-3t}$$

$$\underbrace{9Ce^{-3t} + 4\cdot(-3)Ce^{-3t} + 3Ce^{-3t}}_{=0} = 2e^{-3t}$$

doesn't work!
(because -3 is an eigenvalue)

Try $y(t) = Cte^{-3t}$

$$\frac{dy}{dt} = C(e^{-3t} - 3te^{-3t})$$

$$\frac{d^2y}{dt^2} = C(-3e^{-3t} - 3e^{-3t} + 9te^{-3t})$$

$$= C(-6e^{-3t} + 9te^{-3t})$$

$$C(-6e^{-3t} + 9te^{-3t}) + 4C(e^{-3t} - 3te^{-3t}) + 3Cte^{-3t} = 2e^{-3t}$$

$$Ce^{-3t}(-6 \cancel{+9t} + 4 + \cancel{4\cdot(-3t)} \cancel{+3t}) = 2e^{-3t}$$

$$-2Ce^{-3t} = 2e^{-3t}$$

$$C = -1$$

$\Rightarrow$ particular sol'n

$$y_p(t) = -te^{-3t}$$

Ex Find a particular sol'n to

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}$$

$$s^2 + 2s + 1 = 0$$

$$(s+1)^2 = 0$$

$$s = -1 \quad \text{(repeated)}$$

Try $y(t) = C t^2 e^{-t}$

$$\frac{dy}{dt} = C\left(2te^{-t} - t^2e^{-t}\right)$$

$$\frac{d^2y}{dt^2} = C\left(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2e^{-t}\right)$$

$$= C\left(2e^{-t} - 4te^{-t} + t^2e^{-t}\right)$$

$$C\left(2e^{-t} - 4te^{-t} + t^2e^{-t}\right) + 2C\left(2te^{-t} - t^2e^{-t}\right) + Ct^2e^{-t} = e^{-t}$$

$$Ce^{-t}\left(2 \cancel{- 4t} \cancel{+ t^2} \cancel{+ 4t} \cancel{- 2t^2} \cancel{+ t^2}\right) = e^{-t}$$

$$2Ce^{-t} = e^{-t}$$

$$C = \frac{1}{2} \quad \Rightarrow \text{ particular sol'n}$$

$$y_p(t) = \frac{1}{2}t^2e^{-t}$$

Summarize : to find a particular sol'n to

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = ke^{\alpha t}$$

- If α is not an eigenvalue, try $y(t) = Ce^{\alpha t}$
- If α is a single eigenvalue, try $y(t) = Cte^{\alpha t}$
- If α is a repeated eigenvalue, try $y(t) = Ct^2e^{\alpha t}$

Ex Solve initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = -e^{-t}, \quad y(0)=1, \quad y'(0)=-3.$$

① particular sol'n to inhomo
② general sol'n to homo
$\Rightarrow$ general sol'n to inhomo

③ use initial condition to determine C_1, C_2

$$s^2 + 2s + 5 = 0$$

$$s = \frac{-2 \pm \sqrt{4-4\cdot1\cdot5}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm4i}{2} = -1 \pm 2i$$

① Try $y(t) = Ce^{-t}$

$$\frac{dy}{dt} = -Ce^{-t}$$

$$\frac{d^2y}{dt} = Ce^{-t}$$

$$Ce^{-t} + 2\cdot(-Ce^{-t}) + 5Ce^{-t} = -e^{-t}$$

$$Ce^{-t}(1-2+5) = -e^{-t}$$

$$4Ce^{-t} = -e^{-t}$$

$$C = -\frac{1}{4}$$

$\Rightarrow$ particular sol'n to inhomo.

$$y_p(t) = -\frac{1}{4}e^{-t}$$

② general sol'n to homo.

$$y_h(t) = C_1 \underline{e^{-t}\cos(2t)} + C_2 \underline{e^{-t}\sin(2t)}$$

$\Rightarrow$ general sol'n to inhomo.

$$y(t) = -\frac{1}{4}e^{-t} + C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$$

(3) $y'(t) = \frac{1}{4}e^{-t} + C_1\left(-e^{-t}\cos(2t) - 2e^{-t}\sin(2t)\right)$

$\qquad\qquad + C_2\left(-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)\right)$

initial condition $\Rightarrow$ $y(0) = -\frac{1}{4} + C_1 = 1 \qquad \Rightarrow C_1 = \frac{5}{4}$

$y'(0) = \frac{1}{4} - C_1 + 2C_2 = -3$

$\Rightarrow \frac{1}{4} - \frac{5}{4} + 2C_2 = -3$

$2C_2 = -2$

$C_2 = -1$

$\Rightarrow y(t) = -\frac{1}{4}e^{-t} + \frac{5}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)$

- The same method of finding particular sol'ns also works for complex exponentials

Ex Find (complex) particular sol'n to

$$\frac{d^2y}{dt^2} + y = e^{it} = \cos t + i\sin t$$

$s^2 + 1 = 0 \qquad s = \pm i$

Try $y(t) = Cte^{it}$

$$\frac{dy}{dt} = C(e^{it} + ite^{it})$$

$$\frac{d^2y}{dt^2} = C(ie^{it} + ie^{it} - te^{it})$$

$$= C(2ie^{it} - te^{it})$$

$$C(2ie^{it} - te^{it}) + Cte^{it} = e^{it}$$

$$C e^{it}\left(2i \cancel{-t} \cancel{+t}\right) = e^{it}$$

$$2iCe^{it} = e^{it}$$

$$C = \frac{1}{2i}$$

$\Rightarrow$ particular sol'n $\quad y_p(t) = \frac{1}{2i} t e^{it}$

$$= -\frac{1}{2} i t (\cos t + i \sin t)$$

$$= -\frac{1}{2} i t \cos t + \frac{1}{2} t \sin t$$

$$= \frac{1}{2} t \sin t + i\left(-\frac{1}{2} t \cos t\right)$$

$$\frac{d^2}{dt^2}\left(\frac{1}{2} t \sin t + i\left(-\frac{1}{2} t \cos t\right)\right) + \left(\frac{1}{2} t \sin t + i\left(-\frac{1}{2} t \cos t\right)\right)$$
$$= \cos t + i \sin t$$

real part: $\quad \frac{d^2}{dt^2}\left(\frac{1}{2} t \sin t\right) + \frac{1}{2} t \sin t = \cos t$

$\Rightarrow$ $\frac{1}{2} t \sin t$ is a particular sol'n to

$$\frac{d^2 y}{dt^2} + y = \cos t$$