Ex $\begin{cases} \dfrac{dy_1}{dt} = 4y_1 - 2y_2 \\ \dfrac{dy_2}{dt} = y_1 + y_2 \end{cases}$ Find general sol'n, sketch phase portrait, determine type of equilibrium pt

$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$ $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$$(4-\lambda)(1-\lambda) - (-2)\cdot 1 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

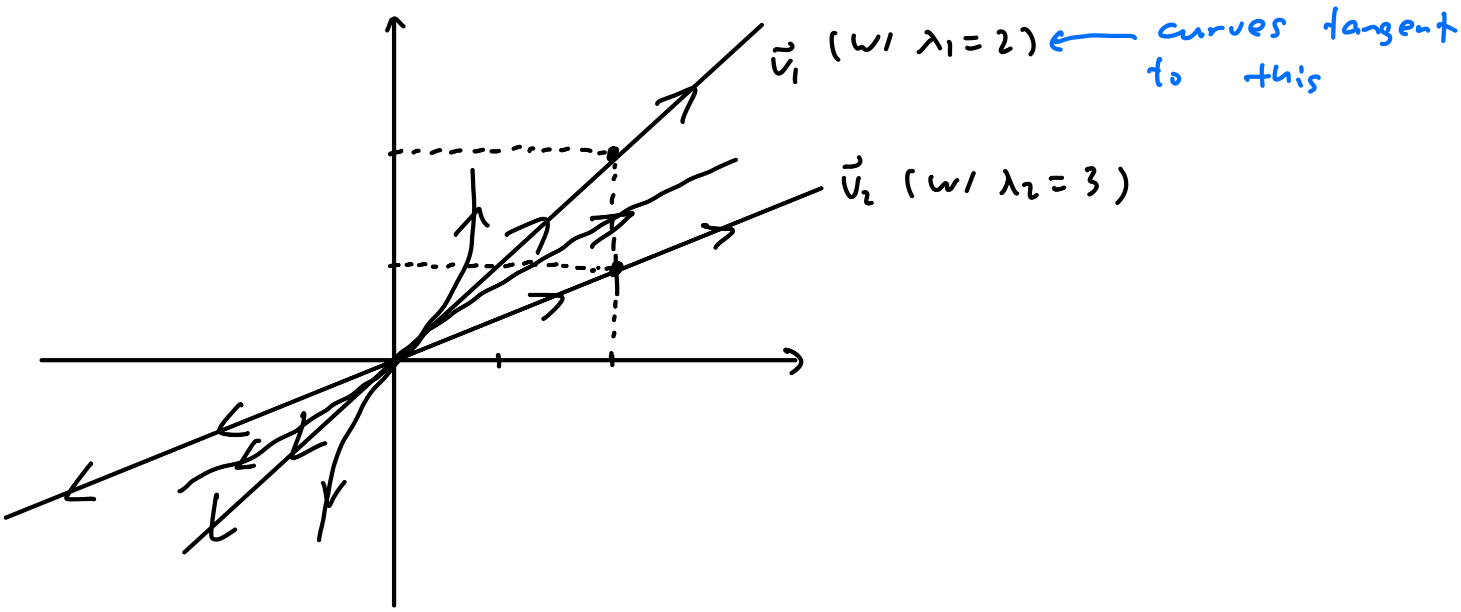
$$(\lambda - 2)(\lambda - 3) = 0$$

$\lambda_1 = 2$, $\lambda_2 = 3$ (type: source)

$\boxed{\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}}$

$\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ General sol'n: $\vec{Y}(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$



Ex $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -1 & -5 \\ 2 & -3 \end{pmatrix} \vec{Y}$

Solve initial value problem w/ $\vec{Y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Sketch phase portrait, determine type.

$$(-1-\lambda)(-3-\lambda) - (-5)\cdot 2 = 0$$

$$3 + 3\lambda + \lambda + \lambda^2 + 10 = 0$$

$$\lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4\cdot 1\cdot 13}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

$\alpha = -2 \quad \beta = 3$

(type: spiral sink)

For $\lambda = -2 + 3i$

$$\vec{v} = \begin{pmatrix} 5 \\ -1-(-2+3i) \end{pmatrix} = \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$$

$\Rightarrow$ complex sol'n $\vec{Y}(t) = e^{(-2+3i)t} \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$

$$= e^{-2t} e^{3it} \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$$

$$= e^{-2t} \left(\cos(3t) + i \sin(3t) \right) \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} \underline{5\cos(3t)} + 5i\sin(3t) \\ \underline{\cos(3t)} - 3i\cos(3t) + i\sin(3t) + \underline{3\sin(3t)} \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} 5\cos(3t) \\ \cos(3t) + 3\sin(3t) \end{pmatrix} + i\, e^{-2t} \begin{pmatrix} 5\sin(3t) \\ -3\cos(3t) + \sin(3t) \end{pmatrix}$$

$\Rightarrow$ General sol'n $\vec{Y}(t) = C_1 e^{-2t}\begin{pmatrix} 5\cos(3t) \\ \cos(3t)+3\sin(3t)\end{pmatrix}$

$$+ C_2 e^{-2t}\begin{pmatrix} 5\sin(3t) \\ -3\cos(3t)+\sin(3t)\end{pmatrix}$$

Initial condition $\Rightarrow$ $C_1\begin{pmatrix}5\\1\end{pmatrix} + C_2\begin{pmatrix}0\\-3\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

$$\begin{cases} 5C_1 = -1 \\ C_1 - 3C_2 = 2 \end{cases}$$

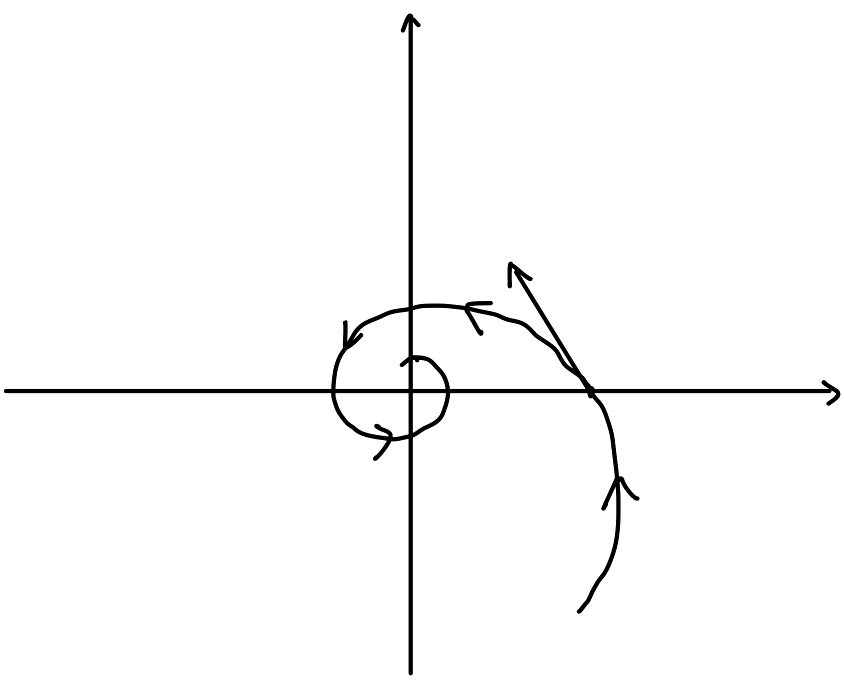
$C_1 = -\frac{1}{5}$

$-\frac{1}{5} - 3C_2 = 2$

$-3C_2 = \frac{11}{5}$

$C_2 = -\frac{11}{15}$

$$\Rightarrow \vec{Y}(t) = -\frac{1}{5}e^{-2t}\begin{pmatrix} 5\cos(3t) \\ \cos(3t)+3\sin(3t)\end{pmatrix} - \frac{11}{15}e^{-2t}\begin{pmatrix} 5\sin(3t) \\ -3\cos(3t)+\sin(3t)\end{pmatrix}$$



$$\begin{pmatrix}-1 & -5\\ 2 & -3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$$

Ex $\quad \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = 0$

$s^2 - s - 6 = 0 \qquad s_1 = 3,\ s_2 = -2$

$y(t) = C_1 e^{3t} + C_2 e^{-2t}$

Convert to first order system.

Find general sol'n to this system, phase portrait, type.

$v = \dfrac{dy}{dt}$

$$\begin{cases} \dfrac{dy}{dt} = v \\ \dfrac{dv}{dt} = v + 6y \end{cases}$$

$\vec{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} \qquad \dfrac{d\vec{Y}}{dt} = A\vec{Y}$

$A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

$\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ v+6y \end{pmatrix}$

$(0-\lambda)(1-\lambda) - 1\cdot 6 = 0$

$-\lambda + \lambda^2 - 6 = 0$

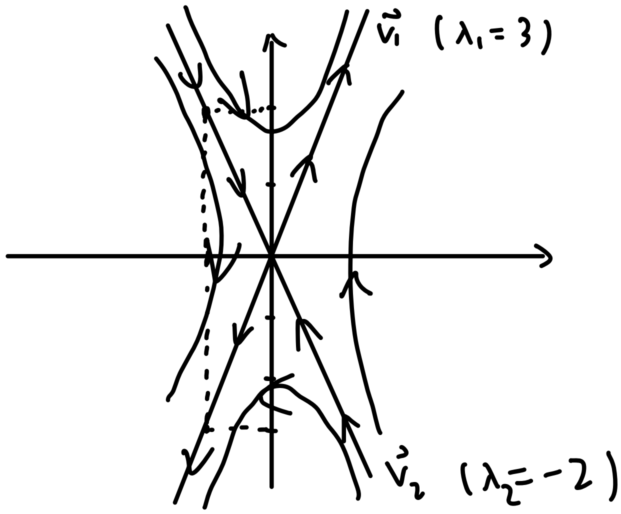
$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$

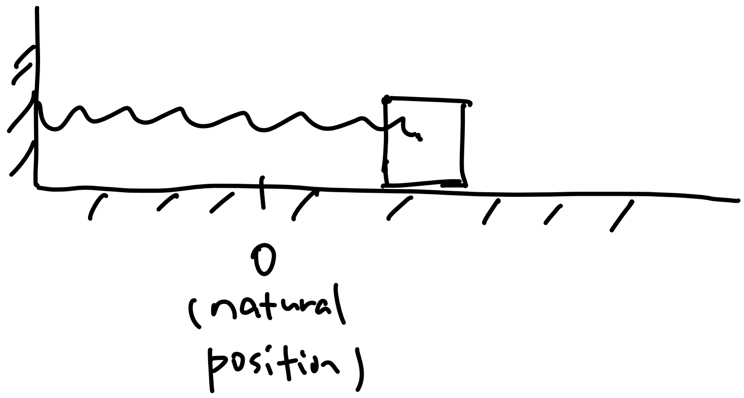
$\lambda_1 = 3 \qquad \lambda_2 = -2 \qquad$ (type: saddle)

$\vec{v}_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\Rightarrow$ General sol'n $\vec{Y}(t) = C_1 e^{3t}\begin{pmatrix} -1 \\ -3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$



4.1 Forced harmonic oscillators



m : mass

k : spring constant

b : damping coefficient

$y(t)$: displacement

$f(t)$: external force

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + f(t)$$

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{f(t)}{m}$$

$$p = \frac{b}{m} \geq 0$$

$$q = \frac{k}{m} > 0$$

$$g(t) = \frac{f(t)}{m}$$

$$\boxed{\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)} \quad \text{------} \quad (1)$$

second order linear DE, inhomogeneous

Let

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0 \quad \text{------} \quad (2)$$

be the associated homogeneous DE.

Recall: The general sol'n to (1) is

$$y(t) = y_p(t) + y_h(t)$$

where y_p is a particular sol'n to (1) and y_h is any sol'n to (2).

We learned how to find $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$:

Solve char. eq. $s^2 + ps + q = 0$

- two distinct real roots s_1, s_2
  $\rightsquigarrow y_1(t) = e^{s_1 t}, \quad y_2(t) = e^{s_2 t}$
- two complex roots $s = \alpha + i\beta$
  $\rightsquigarrow y_1(t) = e^{\alpha t}\cos(\beta t), \quad y_2(t) = e^{\alpha t}\sin(\beta t)$
- repeated root $s = -\frac{p}{2}$
  $\rightsquigarrow y_1(t) = e^{st}, \quad y_2(t) = t e^{st}$

- How to find y_p?

<u>Ex</u> Find a particular sol'n to

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-6t}$$

Try: $y(t) = Ce^{-6t}$

$$36Ce^{-6t} - 24Ce^{-6t} + 3Ce^{-6t} = e^{-6t}$$

$$36C - 24C + 3C = 1$$

$$15C = 1$$

$$C = \frac{1}{15}$$

$\Rightarrow \quad y(t) = \frac{1}{15}e^{-6t}$ is a particular sol'n.