3.5 Special cases (the qualitative behavior of these are NOT IN EXAM)

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad \vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad A: 2\times 2 \text{ matrix}$$

- Repeated eigenvalue λ

To find eigenvectors, we need $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\begin{cases} (a-\lambda)v_1 + bv_2 = 0 \\ cv_1 + (d-\lambda)v_2 = 0 \end{cases}$$

If $a-\lambda, b, c, d-\lambda$ are all zero, then $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$$\begin{cases} \dfrac{dy_1}{dt} = ay_1 \\ \dfrac{dy_2}{dt} = ay_2 \end{cases} \quad \text{fully decoupled}$$

General sol'n $\quad y_1(t) = C_1 e^{at}, \quad y_2(t) = C_2 e^{at}$

$$\vec{Y}(t) = e^{at}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Otherwise, we can only find one eigenvector (or its scalar multiple).

- Try $\vec{Y}(t) = e^{\lambda t}\vec{v}_0 + te^{\lambda t}\vec{v}_1$

$$\frac{d\vec{Y}}{dt} = \lambda e^{\lambda t}\vec{v}_0 + e^{\lambda t}\vec{v}_1 + \lambda t e^{\lambda t}\vec{v}_1$$

$$A\vec{Y} = e^{\lambda t}A\vec{v}_0 + te^{\lambda t}A\vec{v}_1$$

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \quad \Rightarrow$$

$$\lambda e^{\lambda t}\vec{v}_0 + e^{\lambda t}\vec{v}_1 + \lambda t e^{\lambda t}\vec{v}_1 = e^{\lambda t}A\vec{v}_0 + te^{\lambda t}A\vec{v}_1$$

$$\underline{\lambda \vec{v}_0} + \underline{\vec{v}_1} + \underline{\lambda t \vec{v}_1} = \underline{A \vec{v}_0} + \underline{t A \vec{v}_1}$$

$$\Rightarrow \quad \underline{A \vec{v}_1 = \lambda \vec{v}_1}$$

$$\lambda \vec{v}_0 + \vec{v}_1 = A \vec{v}_0 \qquad \underline{\vec{v}_1} = A\vec{v}_0 - \lambda \vec{v}_0 = \underline{(A - \lambda I)\vec{v}_0}$$

Thm If A has repeated eigenvalue λ and A is not in the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, then the general sol'n has the form

$$\vec{Y}(t) = e^{\lambda t} \vec{v}_0 + t e^{\lambda t} \vec{v}_1$$

where $\vec{v}_0$ is any vector, and $\vec{v}_1 = (A - \lambda I)\vec{v}_0$

- If $\vec{v}_1 = (A - \lambda I)\vec{v}_0$ then $A\vec{v}_1 = \lambda \vec{v}_1$ in the above situation

$$(A - \lambda I)\vec{v}_1 = 0$$

Ex Find general sol'n:

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{Y}$$

$$(1-\lambda)(3-\lambda) - (-1)\cdot 1 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$\lambda = 2$ (repeated)

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\vec{v}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\vec{v}_1 = (A - \lambda I)\vec{v}_0 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 \\ c_1 + c_2 \end{pmatrix}$$

$\Rightarrow$ General sol'n $\vec{Y}(t) = e^{2t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + t e^{2t} \begin{pmatrix} -c_1 - c_2 \\ c_1 + c_2 \end{pmatrix}$

- Notice that $\vec{Y}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

Ex For previous example, solve initial value problem w/

$$\vec{Y}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

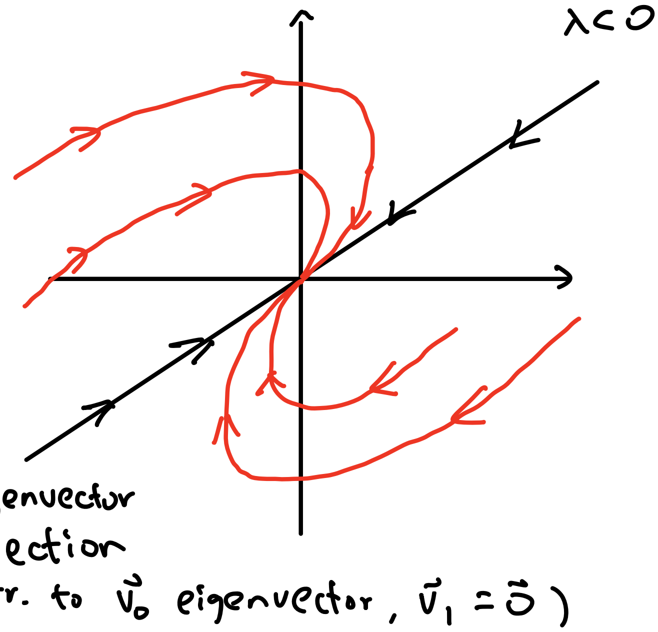
$$\vec{Y}(t) = e^{2t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} + t e^{2t}\begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

Qualitative behavior

$$\vec{Y}(t) = e^{\lambda t}\vec{v}_0 + t e^{\lambda t}\vec{v}_1$$

$\lambda > 0$: source-like

$\lambda < 0$: sink-like



When $t \to \infty$, $\vec{Y}(t) \approx t e^{\lambda t}\vec{v}_1$

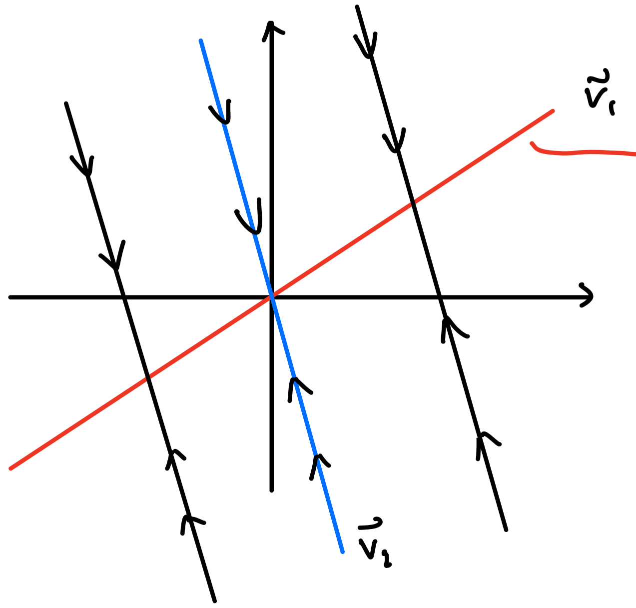
↳ an eigenvector

When $t \to -\infty$, $\vec{Y}(t) \approx t e^{\lambda t}\vec{v}_1$

- When one eigenvalue is $\lambda_1 = 0$
  - If $\lambda_2 \neq 0$, then we have corresponding eigenvectors $\vec{v}_1, \vec{v}_2$

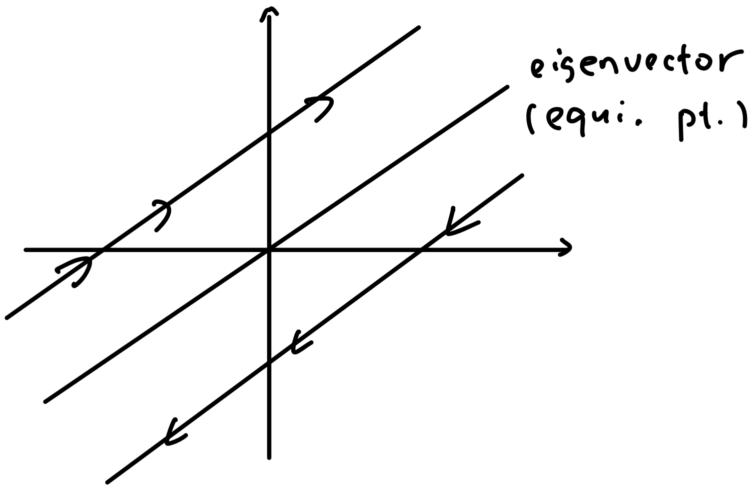
    $\Rightarrow$ General sol'n $\vec{Y}(t) = C_1\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

$\lambda_2 < 0$



any pt on it is an equilibrium pt.

- If $\lambda_2 = 0$.
  - $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
  - Otherwise, $\vec{Y}(t) = \vec{v}_0 + t\,\vec{v}_1$ where $\vec{v}_0$ is arbitrary, $\vec{v}_1 = A\vec{v}_0$



Review sec. 3.3 ~ 3.5

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

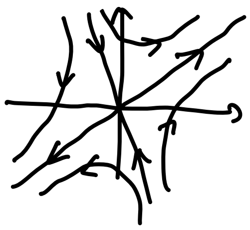
First solve characteristic equation $(a-\lambda)(d-\lambda) - bc = 0$

Case: 2 distinct real eigenvalues λ_1, λ_2

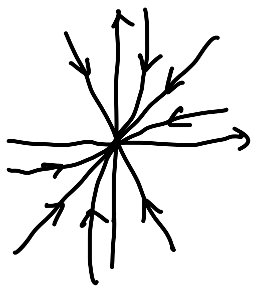
General sol'n: $\vec{Y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

eigenvectors

$\lambda_1 > 0, \lambda_2 < 0$ (saddle)

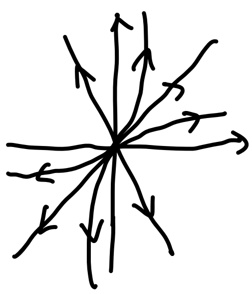


$\lambda_1, \lambda_2 < 0$ (sink)



curves tangent to the straight lines corr. to λ w/ smaller absolute value.

$\lambda_1, \lambda_2 > 0$ (source)



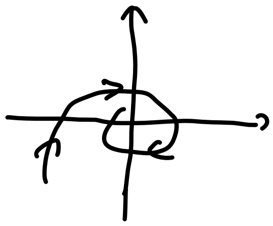
Case: 2 complex eigenvalues $\lambda = \alpha \pm i\beta$

General sol'n: $\vec{Y} = C_1\vec{Y}_1 + C_2\vec{Y}_2$

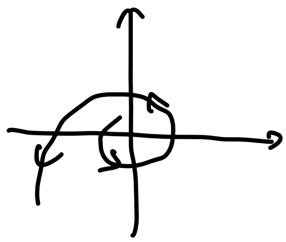
where $\vec{Y}_1, \vec{Y}_2$ are real/imaginary parts of $e^{\lambda t}\vec{v}$

(take one eigenvalue λ and a corr. eigenvector $\vec{v}$).

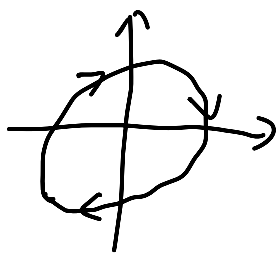
$\alpha < 0$ (spiral sink)



$\alpha > 0$ (spiral source)



$\alpha = 0$ (center)



cw/ccw determined by checking vector field at $(1, 0)$

Case: repeated eigenvalue λ, $A \neq \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

General sol'n: $\vec{Y}(t) = e^{\lambda t} \vec{v}_0 + t e^{\lambda t} \vec{v}_1$

where $\vec{v}_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ is an arbitrary vector, $\vec{v}_1 = (A - \lambda I) \vec{v}_0$

Case: $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

General sol'n: $\vec{Y}(t) = e^{at} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$