3.4 Complex eigenvalue case

Recall: $\frac{d\vec{Y}}{dt} = A\vec{Y}$ $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, A: 2×2 constant matrix

To find special sol'ns like $\vec{Y}(t) = e^{\lambda t}\vec{v}$, we solve characteristic equation

$$(a-\lambda)(d-\lambda) - bc = 0$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Case: two distinct complex roots $\lambda = \alpha \pm i\beta$

Ex $A = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$

$$(0-\lambda)(2-\lambda) - (-1)\cdot 5 = 0$$

$$-2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

For $\lambda = 1+2i$, an eigenvector is $\vec{v} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$

$\Rightarrow$ Complex sol'n $\vec{Y}(t) = e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$

Seperate real/imaginary parts:

$e^{ix} = \cos x + i\sin x$

$$\vec{Y}(t) = e^{t} e^{2it}\begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$$

$$= e^{t}\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$$

$$= e^t \begin{pmatrix} \underline{\cos(2t)} + i\sin(2t) \\ \underline{-\cos(2t)} - i\sin(2t) - 2i\cos(2t) + \underline{2\sin(2t)} \end{pmatrix}$$

$$= \underbrace{e^t \begin{pmatrix} \cos(2t) \\ -\cos(2t) + 2\sin(2t) \end{pmatrix}}_{\text{real part}} + i\, \underbrace{e^t \begin{pmatrix} \sin(2t) \\ -\sin(2t) - 2\cos(2t) \end{pmatrix}}_{\text{imaginary part}}$$

Thm If $\vec{Y}(t)$ is a complex sol'n of $\frac{d\vec{Y}}{dt} = A\vec{Y}$ where A is real, then the real and imaginary parts of $\vec{Y}(t)$ are also sol'ns to this DE.

Apply thm, we get two real sol'ns

$$\vec{Y}_1(t) = e^t \begin{pmatrix} \cos(2t) \\ -\cos(2t) + 2\sin(2t) \end{pmatrix}, \quad \vec{Y}_2(t) = e^t \begin{pmatrix} \sin(2t) \\ -\sin(2t) - 2\cos(2t) \end{pmatrix}$$

Ex Solve initial value problem

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{Y}(t) = C_1 e^t \begin{pmatrix} \cos(2t) \\ -\cos(2t) + 2\sin(2t) \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin(2t) \\ -\sin(2t) - 2\cos(2t) \end{pmatrix}$$

Initial Condition $\Rightarrow$ $C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$$\begin{cases} C_1 = -1 \\ -C_1 - 2C_2 = 3 \end{cases}$$

$$1 - 2C_2 = 3$$
$$-2C_2 = 2$$
$$C_2 = -1$$

$$\Rightarrow \vec{Y}(t) = -e^{t}\begin{pmatrix} \cos(2t) \\ -\cos(2t) + 2\sin(2t) \end{pmatrix} - e^{t}\begin{pmatrix} \sin(2t) \\ -\sin(2t) - 2\cos(2t) \end{pmatrix}$$

Summarize (complex eigenvalues, to get two real special sol'ns)

- Find complex eigenvalues $\lambda = \alpha \pm i\beta$
- For one eigenvalue, say, $\lambda = \alpha + i\beta$, find complex eigenvector $\vec{v}$.
- Write complex sol'n $\vec{Y}(t) = e^{(\alpha+i\beta)t}\vec{v}$ and separate into real / imaginary parts. Each gives a real sol'n.

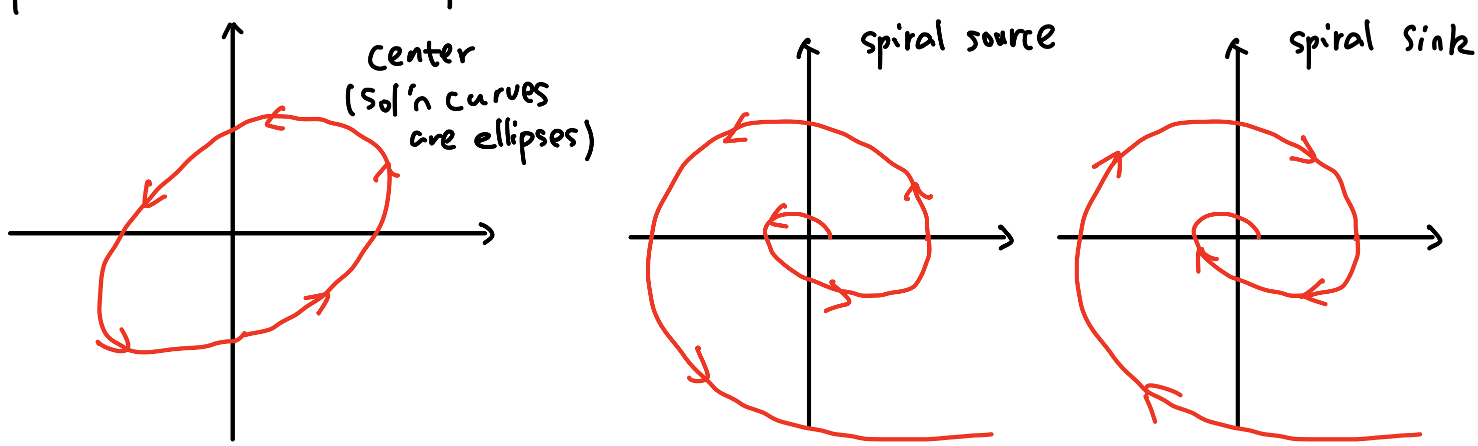
Qualitative behavior

Any real sol'n has the form

$$\vec{Y}(t) = e^{\alpha t} \underbrace{\begin{pmatrix} \_\_ \cos(\beta t) + \_\_ \sin(\beta t) \\ \_\_ \cos(\beta t) + \_\_ \sin(\beta t) \end{pmatrix}}_{\text{rotation around } O \text{ w/ natural period } \frac{2\pi}{\beta}}$$

exponential growth/decay

$$\begin{cases} \alpha > 0 : \text{exponential growth (spiral source)} \\ \alpha < 0 : \text{exponential decay (spiral sink)} \\ \alpha = 0 : \text{sol'n are periodic, rotating around } O \text{ (center)} \end{cases}$$



$\underline{\text{Ex}}$ Sketch phase portrait

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \vec{Y}$$

$$(1-\lambda)(-3-\lambda) - (-5)\cdot 1 = 0$$

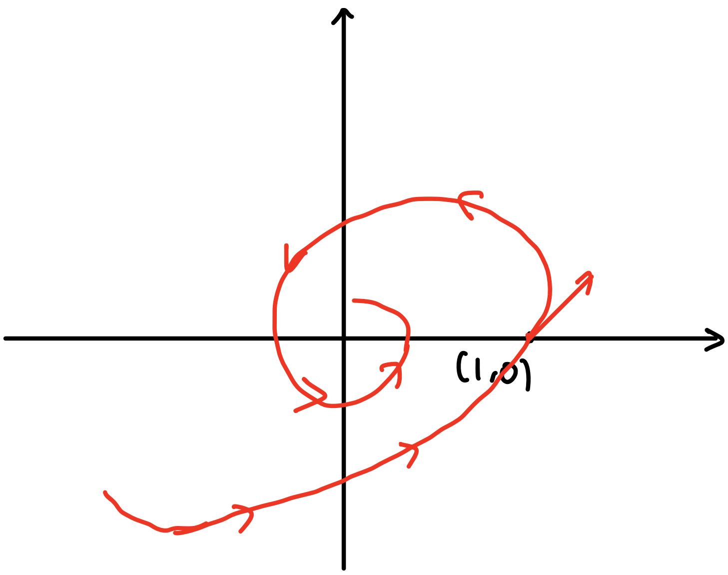
$$-3 - \lambda + 3\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\alpha = -1$ $\quad \beta = 1$

↳ spiral sink



To determine cw/ccw :

Check vector field at (1,0)

$$\begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\text{Ex}}$ Sketch phase portrait

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix} \vec{Y}$$

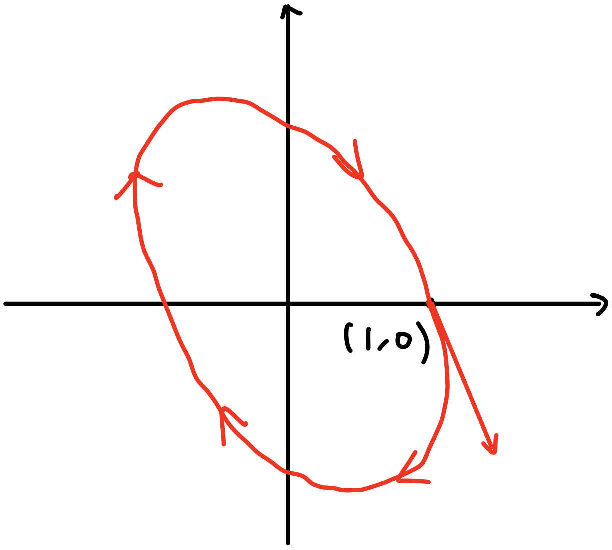
$$(2-\lambda)(-2-\lambda) - 1\cdot(-8) = 0$$

$$-4 - 2\lambda + 2\lambda + \lambda^2 + 8 = 0$$

$$\lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$$

$$\lambda^2 = -4 \qquad \alpha = 0 \quad \beta = 2$$

↳ center



$$\begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

Why ellipses for a center?

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}\vec{Y}$$

$$(0-\lambda)(0-\lambda) - 4\cdot(-1) = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

For $\lambda = 2i$, $\vec{v} = \begin{pmatrix} -4 \\ -2i \end{pmatrix}$

$$\vec{Y}(t) = e^{2it}\begin{pmatrix} -4 \\ -2i \end{pmatrix}$$

$$= (\cos 2t + i\sin 2t)\begin{pmatrix} -4 \\ -2i \end{pmatrix}$$

$$= \begin{pmatrix} -4\cos 2t - 4i\sin 2t \\ -2i\cos 2t + 2\sin 2t \end{pmatrix}$$

$$= \underbrace{\boxed{\begin{pmatrix} -4\cos 2t \\ 2\sin 2t \end{pmatrix}}}_{\vec{Y}_1(t)} + i\underbrace{\boxed{\begin{pmatrix} -4\sin 2t \\ -2\cos 2t \end{pmatrix}}}_{\vec{Y}_2(t)}$$

$$\vec{Y}_1(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -4\cos 2t \\ 2\sin 2t \end{pmatrix}$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \qquad \text{ellipse}$$