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Total time: 75 minutes.
Total points: 100.

Problem 1 ($10 \times 3 = 30$ points). Find the general solution to the following differential equations:

$$(1) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sin t}{y^2}$$

$$y^2 \,\mathrm{d}y = \sin t \,\mathrm{d}t$$

$$\frac{1}{3}y^3 = -\cos t + C$$

$$y^3 = 3(-\cos t + C)$$

$$y = (3(-\cos t + C))^{1/3}$$

$$(2) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{t}y + e^{t^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{1}{t}y = e^{t^2}$$

$$\mu(t) = e^{\int \frac{1}{t}\,\mathrm{d}t} = e^{\ln t} = t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(ty) = te^{t^2}$$

$$ty = \int te^{t^2}\,\mathrm{d}t = \frac{1}{2}\int e^u \,\mathrm{d}u = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

(substitution $u = t^2,\ \mathrm{d}u = 2t\,\mathrm{d}t$)

$$y = \frac{1}{t}(\frac{1}{2}e^{t^2} + C)$$

$$(3) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y^2 t - 2y^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2(t-2)$$

$$\frac{1}{y^2}\,\mathrm{d}y = (t-2)\,\mathrm{d}t$$

(except for a special solution $y(t) = 0$)

$$-\frac{1}{y} = \frac{1}{2}t^2 - 2t + C$$

$$\frac{1}{y} = -(\frac{1}{2}t^2 - 2t + C)$$

$$y = -\frac{1}{\frac{1}{2}t^2 - 2t + C}$$

Therefore the general solution is

$$y = -\frac{1}{\frac{1}{2}t^2 - 2t + C} \text{ or } y = 0$$

Problem 2 ($10 \times 2 = 20$ points). Solve the following initial value problems:

$$(1) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y^2}{t^3}, \quad y(1) = 2$$

$$\frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}t}{t^3}$$

(Here when dividing by y^2, we excluded the case $y = 0$ which is a special solution to the DE, but this does not matter because it does not agree with the initial condition.)

$$-\frac{1}{y} = -\frac{1}{2t^2} + C$$

Using the initial condition $y(1) = 2$, we get

$$-\frac{1}{2} = -\frac{1}{2} + C$$
$$C = 0$$

Therefore

$$-\frac{1}{y} = -\frac{1}{2t^2}$$
$$\frac{1}{y} = \frac{1}{2t^2}$$
$$y = 2t^2$$

$$(2) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty + e^{-t^2} = 0, \quad y(0) = -3$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2ty = -e^{-t^2}$$
$$\mu(t) = e^{\int 2t\,\mathrm{d}t} = e^{t^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{t^2} y) = -1$$
$$e^{t^2} y = -t + C$$

Using the initial condition $y(0) = -3$, we get

$$-3 = 0 + C$$

$$C = -3$$

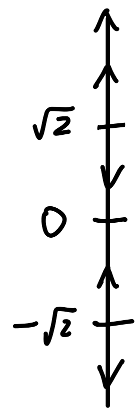
Therefore

$$e^{t^2} y = -t - 3$$
$$y = (-t - 3)e^{-t^2}$$

Problem 3 ($10 \times 2 = 20$ points). Consider the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - 2y$$

(1) Sketch the phase line for this differential equation.
To get equilibrium points, we solve $y^3 - 2y = y(y^2 - 2) = 0$ and get $y = 0, \pm\sqrt{2}$.
Then on the 4 intervals separated by equilibrium points, we test special values $y = -2, -1, 1, 2$ and see that the signs of $y^3 - 2y$ are $-, +, -, +$.



(2) Impose the initial condition $y(1) = -1$. Use Euler's method with $\Delta t = 1/4$ to solve this initial value problem and approximate $y(1.5)$.

$$t_0 = 1, \quad t_1 = 1.25, \quad t_2 = 1.5$$

Therefore we need 2 iterations.

$$y_1 = y_0 + \frac{1}{4} f(t_0, y_0) = -1 + \frac{1}{4}((-1)^3 - 2 \cdot (-1)) = -\frac{3}{4}$$

$$y_2 = y_1 + \frac{1}{4} f(t_1, y_1) = -\frac{3}{4} + \frac{1}{4}((-\frac{3}{4})^3 - 2 \cdot (-\frac{3}{4})) = -\frac{123}{256} \approx -0.48$$

(you may leave either the fraction or a reasonable decimal approximation as the final answer)

Problem 4 ($8+7=15$ points). Consider the damped harmonic oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 0$$

(1) Find two solutions of the form $y(t) = e^{st}$.
The characteristic equation is

$$s^2 + 4s + 3 = (s+1)(s+3) = 0$$

It has roots $s_1 = -1, s_2 = -3$. Therefore we get two particular solutions

$$y_1(t) = e^{-t}, \quad y_2(y) = e^{-3t}$$

(2) Use your result in (1) to solve the initial value problem with initial condition

$$y(0) = 3,\ y'(0) = -7$$

Consider a solution of the form

$$y(t) = C_1 e^{-t} + C_2 e^{-3t}$$
$$y'(t) = -C_1 e^{-t} - 3C_2 e^{-3t}$$

To match with initial condition,

$$C_1 + C_2 = 3, \quad -C_1 - 3C_2 = -7$$

Solve to get

$$C_1 = 1, \quad C_2 = 2$$

Therefore the solution is

$$y(t) = e^{-t} + 2e^{-3t}$$

Problem 5 (15 points). A vat initially contains 4 gallons of salty water with concentration 1 oz/gallon. Salty water with concentration 2 oz/gallon starts to flow into the vat at a rate of 8 gallon/minute, while well-mixed liquid flows out at the same rate. What is the amount of salt in the vat after 10 minutes?

The total volume of liquid in the vat is unchanged. Let $S(t)$ be the amount of salt in the vat. Then the concentration in the vat is $\frac{S}{4}$. We have the DE

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2 \cdot 8 - \frac{S}{4} \cdot 8 = 16 - 2S$$

with initial condition

$$S(0) = 4 \cdot 1 = 4$$

To solve this DE,

$$\frac{\mathrm{d}S}{16 - 2S} = \mathrm{d}t$$

$$-\frac{1}{2} \ln|16 - 2S| = t + C$$

Using the initial condition, we get

$$-\frac{1}{2} \ln|16 - 2 \cdot 4| = 0 + C$$

$$C = -\frac{1}{2} \ln 8$$

This also tells us that $\ln|16 - 2S|$ can be replaced by $\ln(16 - 2S)$ because $16 - 2S$ is positive initially.

$$-\frac{1}{2} \ln(16 - 2S) = t - \frac{1}{2} \ln 8$$

$$\ln(16 - 2S) = -2t + \ln 8$$

$$16 - 2S = e^{-2t + \ln 8} = 8e^{-2t}$$

$$2S = 16 - 8e^{-2t}$$

$$S = 8 - 4e^{-2t}$$

After 10 minutes, that is, $t = 10$, the amount of salt in the vat is

$$S(10) = 8 - 4e^{-20}$$