EXAM: class average = 77

3.3 (continued)

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad A: 2\times 2 \text{ matrix}$$

Consider case: 2 distinct real eigenvalues, nonzero

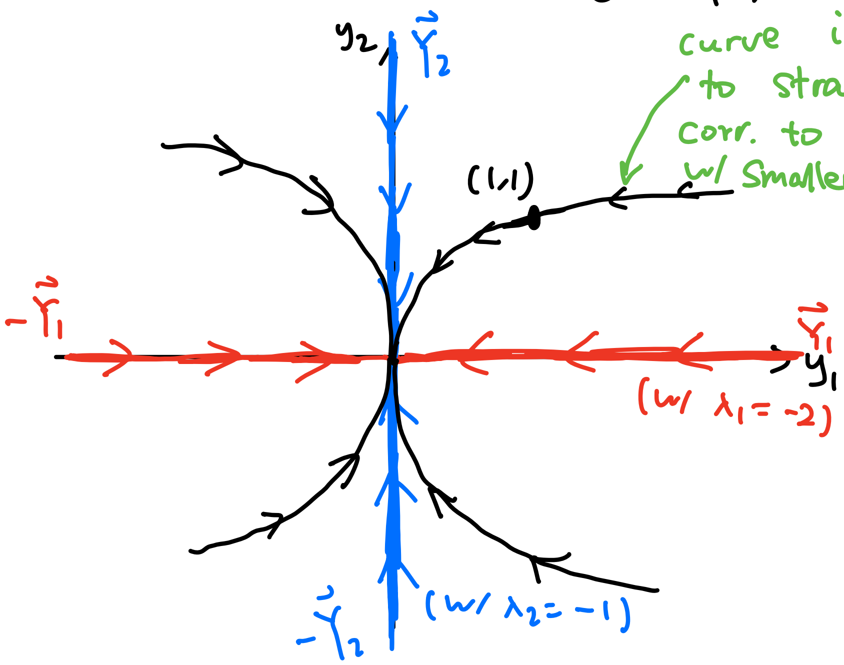
Case: one eigenvalue >0, another <0 (saddle) (done)

Case: both eigenvalues <0 (sink)

Consider $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\vec{Y}$

$\lambda_1 = -2$, $\lambda_2 = -1$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\vec{Y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{Y}_2(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Consider sol'n w/ initial condition

$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{Y}(t) = C_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

initial $\Rightarrow C_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

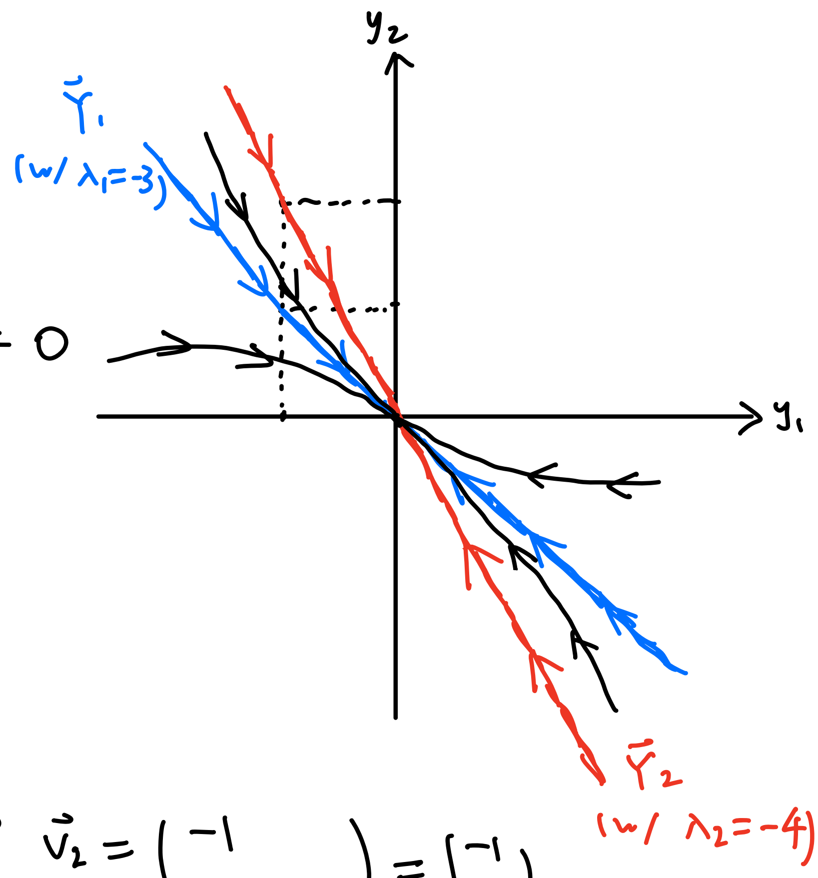
$C_1 = 1$, $C_2 = 1$

$\vec{Y}(t) = e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ e^{-t} \end{pmatrix}$

As $t \to \infty$, both e^{-2t} and $e^{-t} \to 0$, while e^{-2t} is much smaller than e^{-t}

Ex Sketch phase portrait:

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ -2 & -5 \end{pmatrix} \vec{Y}$$



$$(-2-\lambda)(-5-\lambda) - 1\cdot(-2) = 0$$

$$\lambda^2 + 7\lambda + 10 + 2 = 0$$

$$\lambda^2 + 7\lambda + 12 = 0$$

$$(\lambda+3)(\lambda+4) = 0$$

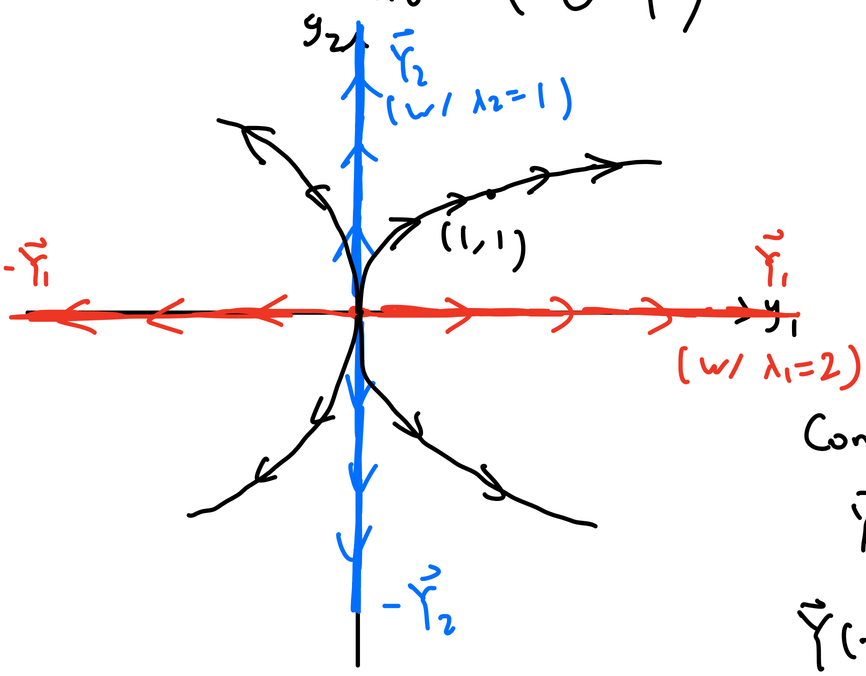
$$\lambda_1 = -3, \quad \lambda_2 = -4$$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ -2-(-3) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ -2-(-4) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\vec{Y}_1(t) = e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \vec{Y}_2(t) = e^{-4t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Case: both eigenvalues > 0 (source)

Consider $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \vec{Y}$



$$\lambda_1 = 2 \qquad \lambda_2 = 1$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{Y}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{Y}_2(t) = e^{t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider sol'n w/ initial condition

$$\vec{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{Y}(t) = \begin{pmatrix} e^{2t} \\ e^{t} \end{pmatrix}$$

As $t \to -\infty$, both e^{2t} and $e^{t} \to 0$, while e^{t} is much larger than e^{2t}

<u>Ex</u> Sketch phase portrait:

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \vec{Y}$$

$$(3-\lambda)(4-\lambda) - 2 \times 1 = 0$$

$$\lambda^2 - 7\lambda + 12 - 2 = 0$$

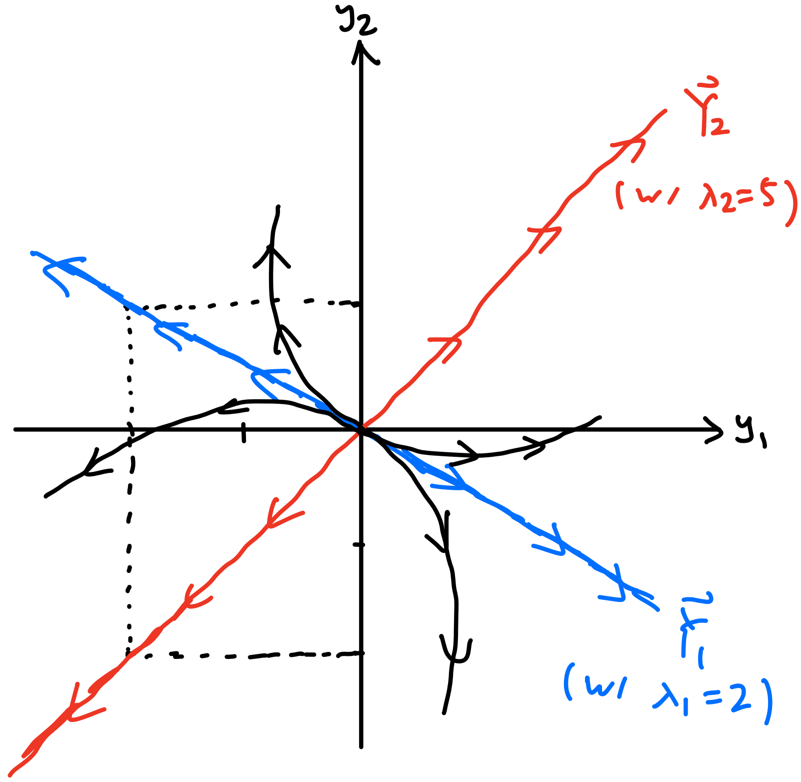
$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 5$$

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

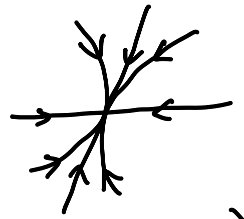
$$\vec{Y}_1(t) = e^{2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \vec{Y}_2(t) = e^{5t}\begin{pmatrix} -2 \\ -2 \end{pmatrix}$$



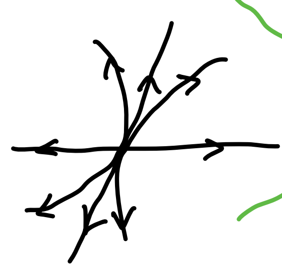
<u>Summary</u> (case: 2 distinct real eigenvalues, nonzero)

- One eigenvalue > 0, another < 0 (saddle)

- Both eigenvalues < 0 (sink)



- Both eigenvalues > 0 (source)



curves tan. to straight line w/ eigenvalue w/ smaller abs. value.

- Sink is stable because any sol'n starting from nearby (0,0) will converge to (0, 0) as $t \to \infty$
- Source and saddle are unstable because there exists sol'n starting from nearby (0,0) that gets away as $t \to \infty$.