3.3 Real eigenvalue case

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad \vec{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad A \text{ is } n\times n \text{ constant matrix}$$

Recall:

- To find sol'ns like $\vec{Y}(t) = e^{\lambda t}\vec{v}$, we need $\vec{v} \neq \vec{0}$

$$A\vec{v} = \lambda\vec{v}$$

eigenvalue (λ)

eigenvector ($\vec{v}$) corresponding to λ.

- When $n=2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, to find eigenvalues of A, solve characteristic eq.

$$(a-\lambda)(d-\lambda) - bc = 0$$

- For each eigenvalue λ, to find eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, need

$$\begin{cases} (a-\lambda)v_1 + bv_2 = 0 \\ cv_1 + (d-\lambda)v_2 = 0 \end{cases}$$

Take $\vec{v} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ (as long as it's not $\vec{0}$)

When solving char. eq., possible outcomes:

$$\begin{cases} \boxed{2 \text{ distinct real eigenvalues (sec. 3.3)}} \\ 2 \text{ distinct complex roots (sec. 3.4)} \\ 1 \text{ multiple real root (sec. 3.5)} \end{cases}$$

assuming this case here!

Assume we get two distinct real eigenvalues λ_1, λ_2, w/ corresponding eigenvectors $\vec{v}_1, \vec{v}_2$, then we get two special sol'ns:

$$\vec{Y}_1(t) = e^{\lambda_1 t}\vec{v}_1 \quad , \quad \vec{Y}_2(t) = e^{\lambda_2 t}\vec{v}_2$$

$\Rightarrow$ General sol'n: $\vec{Y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$.

Ex Solve initial value problem

$$\begin{cases} \dfrac{dy_1}{dt} = -y_2 \\ \dfrac{dy_2}{dt} = -2y_1 + y_2 \end{cases} \qquad \begin{cases} y_1(0) = -1 \\ y_2(0) = 3 \end{cases}$$

$$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} \qquad \vec{Y}(0) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$(0-\lambda)(1-\lambda) - (-1)(-2) = 0$$

$$-\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\underbrace{\lambda_1 = -1}, \quad \underbrace{\lambda_2 = 2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ Special sol'ns: $\vec{Y}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{Y}_2(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

General sol'n: $\vec{Y}(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Initial condition $\Rightarrow$ $C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$$\begin{cases} C_1 + C_2 = -1 \\ C_1 - 2C_2 = 3 \end{cases} \qquad -3C_2 = 4 \qquad C_2 = -\frac{4}{3}$$

$$C_1 = -1 - C_2 = -1 + \frac{4}{3} = \frac{1}{3}$$

$$\Rightarrow \vec{Y}(t) = \frac{1}{3} e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{4}{3} e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

- Behavior of equilibrium pt $(0,0)$
  (in case "2 distinct real eigenvalues")

$$\begin{cases} \text{One eigenvalue} > 0, \text{ another} < 0 \\ \text{Both eigenvalues} < 0 \\ \text{Both eigenvalues} > 0 \\ \text{Some eigenvalue} = 0 \text{ (leave to sec. 3.5)} \end{cases}$$

<u>Case "One eigenvalue >0, another <0"</u> (saddle)

Say, $\lambda_1 > 0$, $\lambda_2 < 0$

Consider $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \vec{Y}$

$$(2-\lambda)(-1-\lambda) - 0 \cdot 0 = 0$$

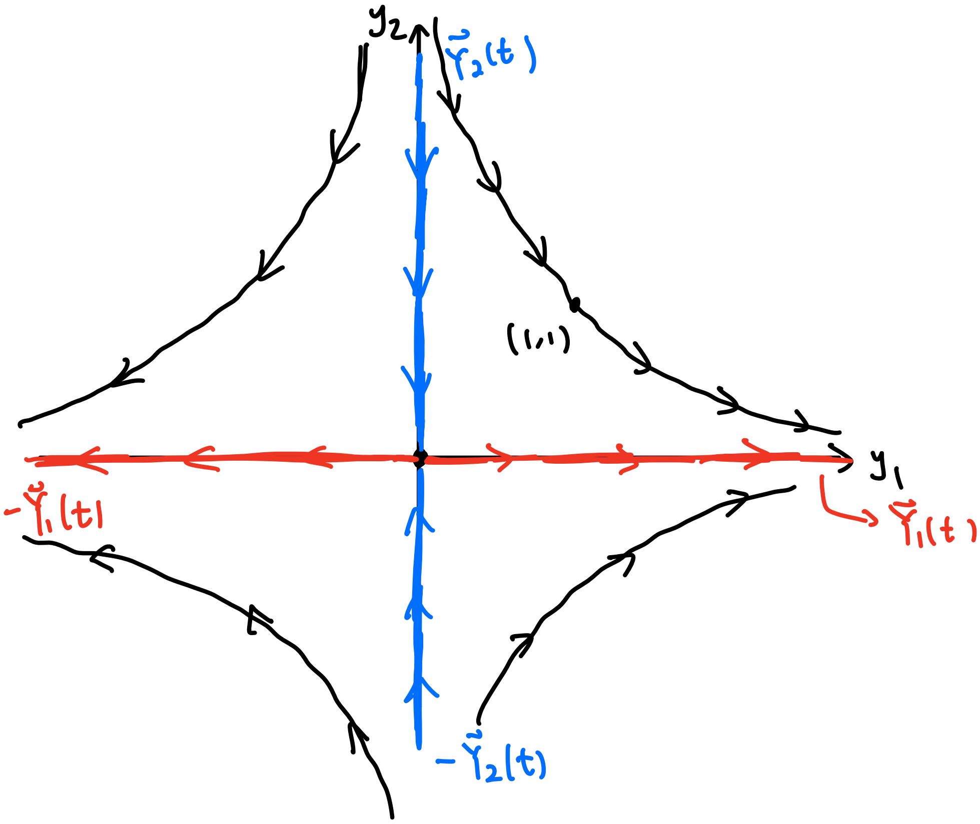
$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = -1$$

$$\begin{cases} (a-\lambda)v_1 + b v_2 = 0 \\ c v_1 + (d-\lambda) v_2 = 0 \end{cases}$$

$$\vec{v}_1 = \begin{pmatrix} d-\lambda \\ -c \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{Y}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{Y}_2(t) = e^{-t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

General sol'n: $\vec{Y}(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



Consider the sol'n w/ initial condition $\vec{Y}(0) = \begin{pmatrix}1\\1\end{pmatrix}$

$$C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow C_1 = 1,\ C_2 = 1$$

$$\vec{Y}(t) = e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + e^{-t}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}e^{2t}\\e^{-t}\end{pmatrix}$$

- In case of a saddle, most sol'ns get away from $(0,0)$ as $t \to \infty$ (unless the initial pt is exactly on the straight line corresponding to the negative eigenvalue).
- To sketch phase portrait of a general saddle:
  - Sketch straight line sol'ns $\vec{Y}_1, \vec{Y}_2$
  - Sketch curves in the 4 regions cut by the lines, and draw arrows.

Ex Draw phase portrait of previous example.

Special sol'ns : $\vec{Y}_1(t) = e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$, $\vec{Y}_2(t) = e^{2t}\begin{pmatrix}1\\-2\end{pmatrix}$