3.2 Straight line solutions

Consider $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$ where $\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$,

(linear, homogeneous, constant coefficient)

A is an $n \times n$ constant matrix

$$\frac{d\vec{Y}}{dt} = A(t)\vec{Y} + \vec{b}(t)$$

Try to find special sol'ns of the form

$\vec{Y}(t) = e^{\lambda t}\vec{v}$ where λ is a constant, $\vec{v}$ is a constant vector.

$$\frac{d\vec{Y}}{dt} = \lambda e^{\lambda t}\vec{v}$$

$$\Rightarrow \lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$$

$$A\vec{v} = \lambda\vec{v}$$

Recall: Let A be an $n \times n$ matrix. If

$$A\vec{v} = \lambda\vec{v}$$

for some constant λ and nonzero vector $\vec{v}$, then we say λ is an eigenvalue of A and $\vec{v}$ is an eigenvector of A corresponding to λ.

- To find eigenvalues of A

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$

Case $n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$A\vec{v} = \lambda \vec{v}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} a v_1 + b v_2 = \lambda v_1 \\ c v_1 + d v_2 = \lambda v_2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \underline{(a-\lambda)} v_1 + \underline{b} v_2 = 0 \\ \underline{c} v_1 + \underline{(d-\lambda)} v_2 = 0 \end{cases}$$

$2v_1 + 3v_2 = 0$

$4v_1 + 6v_2 = 0$

$\begin{cases} v_1 = -3 \\ v_2 = 2 \end{cases}$

- If one eq. is in proportion w/ the other, then one can find nonzero sol'ns $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Need $\dfrac{a-\lambda}{c} = \dfrac{b}{d-\lambda}$ $\rightsquigarrow$ $(a-\lambda)(d-\lambda) = bc$

$$\boxed{(a-\lambda)(d-\lambda) - bc = 0}$$

characteristic equation.

Summarize for $A\vec{v} = \lambda\vec{v}$ (case $n=2$):

- Find eigenvalues λ by solving characteristic equation.
- To find $\vec{v}$ for an eigenvalue λ, one needs

$(a-\lambda)v_1 + b v_2 = 0$ $\begin{cases} v_1 = -b \\ v_2 = a-\lambda \end{cases}$

(when $a-\lambda$, b are not both zero).

Ex Find sol'ns like $\vec{Y}(t) = e^{\lambda t}\vec{v}$ for

$$\begin{cases} \dfrac{dy_1}{dt} = -2y_1 + 3y_2 \\ \dfrac{dy_2}{dt} = y_1 \end{cases}$$

$$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad A = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$$

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

- Find eigenvalues by char. eq. $\boxed{(a-\lambda)(d-\lambda) - bc = 0}$

$$(-2-\lambda)(-\lambda) - 3 = 0$$

$$2\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 + 2\lambda - 3 = 0$$

$$(\lambda+3)(\lambda-1) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = 1$$

$$\boxed{\begin{cases} v_1 = -b \\ v_2 = a - \lambda \end{cases}}$$

- For $\lambda_1 = -3$,

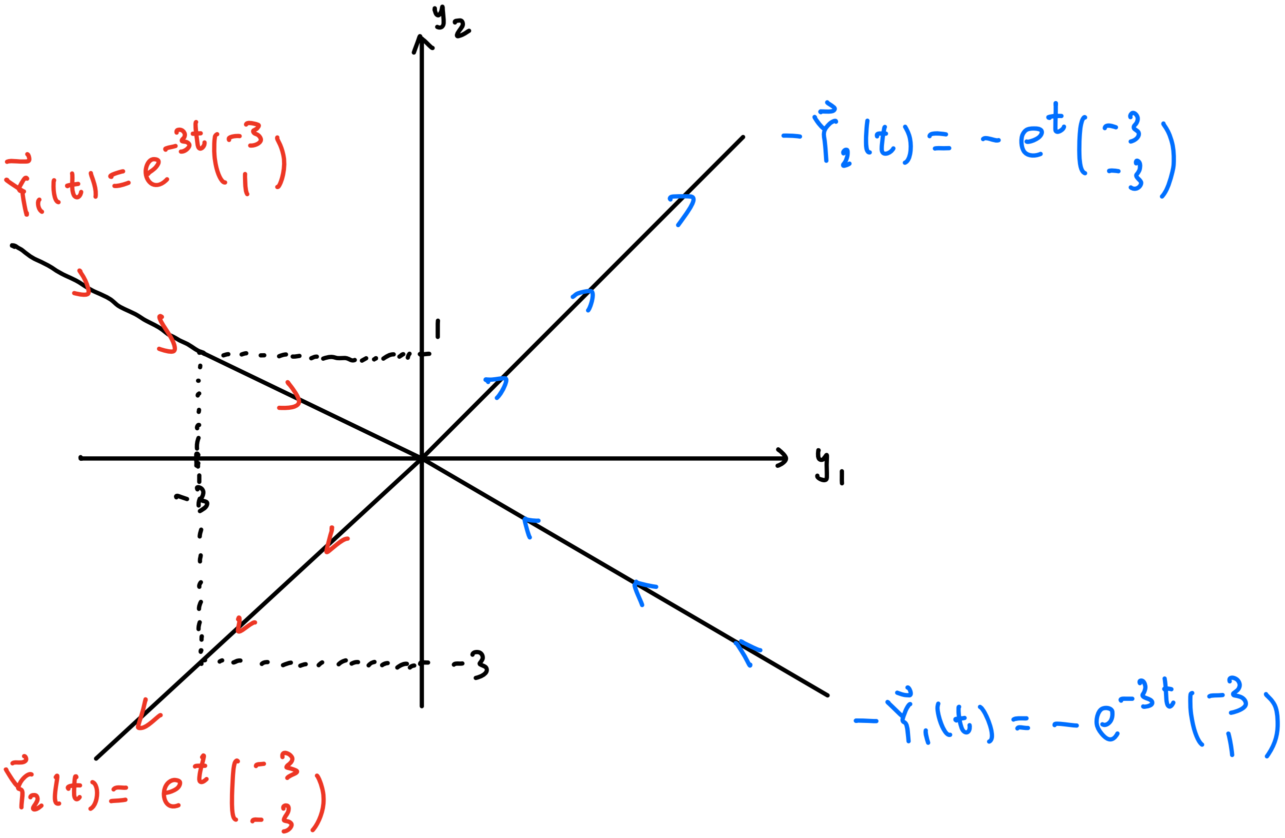
$$\vec{v} = \begin{pmatrix} -3 \\ -2-(-3) \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

- For $\lambda_2 = 1$,

$$\vec{v} = \begin{pmatrix} -3 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

$\Rightarrow$ Special sol'ns $\vec{Y}_1(t) = e^{-3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\vec{Y}_2(t) = e^{t}\begin{pmatrix} -3 \\ -3 \end{pmatrix}.$$



- If we find 2 special sol'ns to $\frac{d\vec{Y}}{dt} = A\vec{Y}$ (case $n=2$), say, $\vec{Y}_1(t)$, $\vec{Y}_2(t)$, which are not constant multiples of each other, then the general sol'n is

$$\vec{Y}(t) = C_1\vec{Y}_1(t) + C_2\vec{Y}_2(t).$$

Ex In previous example, solve initial value problem w/

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = -2 \end{cases}$$

$$\vec{Y}(t) = C_1 e^{-3t}\begin{pmatrix}-3\\1\end{pmatrix} + C_2 e^{t}\begin{pmatrix}-3\\-3\end{pmatrix}$$

initial condition: $\vec{Y}(0) = \begin{pmatrix}1\\-2\end{pmatrix}$

$$C_1\begin{pmatrix}-3\\1\end{pmatrix} + C_2\begin{pmatrix}-3\\-3\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$$

$$\begin{cases} -3C_1 - 3C_2 = 1 \\ C_1 - 3C_2 = -2 \end{cases}$$

$-12C_2 = -5 \qquad C_2 = \frac{5}{12}$

$C_1 - 3 \times \frac{5}{12} = -2$

$C_1 = \frac{5}{4} - 2 = -\frac{3}{4}$

$$\Rightarrow \vec{Y}(t) = -\frac{3}{4} e^{-3t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \frac{5}{12} e^{t} \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

Ex Find general sol'n to

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = x - y \end{cases}$$

$\vec{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

- Find eigenvalues by char. eq.

$$(1-\lambda)(-1-\lambda) - 1 = 0$$

$$-1 \cancel{-\lambda} \cancel{+\lambda} + \lambda^2 - 1 = 0$$

$$\lambda^2 - 2 = 0$$

$$\lambda_1 = \sqrt{2}, \quad \lambda_2 = -\sqrt{2}$$

$\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$

- For $\lambda_1 = \sqrt{2}$,

$\vec{v} = \begin{pmatrix} -1 \\ 1-\sqrt{2} \end{pmatrix}$

- For $\lambda_2 = -\sqrt{2}$,

$\vec{v} = \begin{pmatrix} -1 \\ 1+\sqrt{2} \end{pmatrix}$

$\Rightarrow$ Special sol'ns $\vec{Y}_1(t) = e^{\sqrt{2}t} \begin{pmatrix} -1 \\ 1-\sqrt{2} \end{pmatrix}, \quad \vec{Y}_2(t) = e^{-\sqrt{2}t} \begin{pmatrix} -1 \\ 1+\sqrt{2} \end{pmatrix}$

$\Rightarrow$ General sol'n $\vec{Y}(t) = C_1 e^{\sqrt{2}t} \begin{pmatrix} -1 \\ 1-\sqrt{2} \end{pmatrix} + C_2 e^{-\sqrt{2}t} \begin{pmatrix} -1 \\ 1+\sqrt{2} \end{pmatrix}$