2.7 The SIR model of an epidemic

Consider a population and a disease is spreading.

$$\begin{cases} S(t): \text{ fraction of pop. that can catch the disease} \\ I(t): \text{ fraction of pop. that has the disease and can spread it} \\ R(t): \text{ fraction of pop. that has recovered and cannot catch it again.} \end{cases}$$

$$S(t) + I(t) + R(t) = 1$$

$$\begin{cases} \dfrac{dS}{dt} = -\alpha S I \\ \dfrac{dI}{dt} = \alpha S I - \beta I \\ \dfrac{dR}{dt} = \beta I \end{cases} \qquad \alpha > 0, \ \beta > 0$$

- $S(t)$ and $I(t)$ satisy a system of 2 DEs, and $R(t)$ can be obtained by $R = 1 - S - I$

$$\begin{cases} \dfrac{dS}{dt} = -\alpha S I \\ \dfrac{dI}{dt} = \alpha S I - \beta I = I(\alpha S - \beta) \end{cases}$$

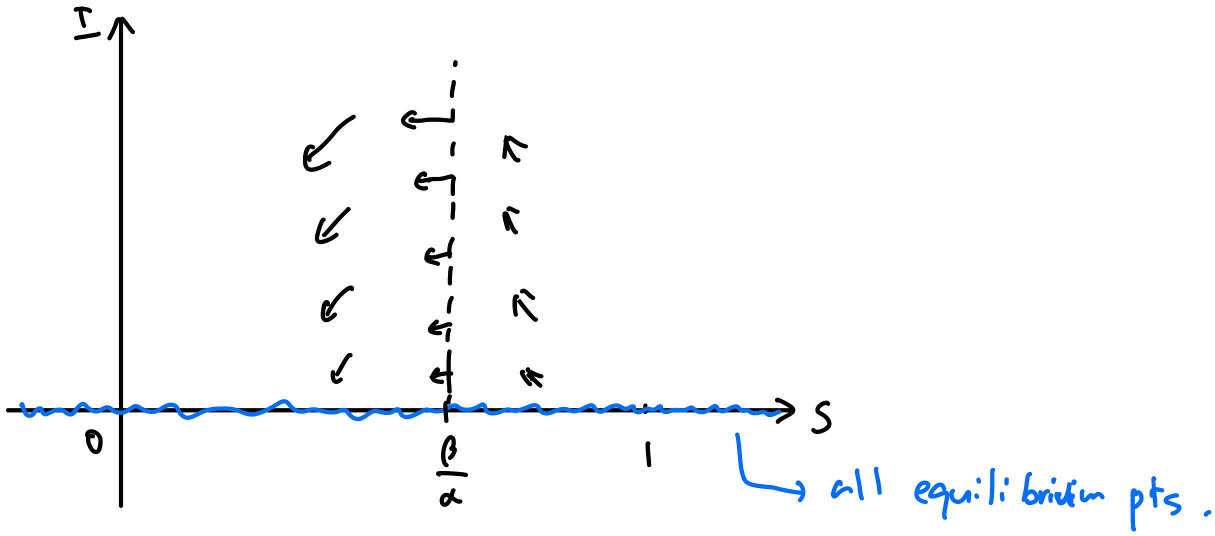
Equilibrium pts

$$\begin{cases} -\alpha S I = 0 \\ \alpha S I - \beta I = 0 \Rightarrow I(\alpha S - \beta) = 0 \end{cases}$$

If $I = 0$, S arbitrary.

If $I \neq 0$, then $S = 0$, $S = \dfrac{\beta}{\alpha}$ contradiction $\Rightarrow$ no sol'n

Equilibrium pts : $I = 0$, S arbitrary.

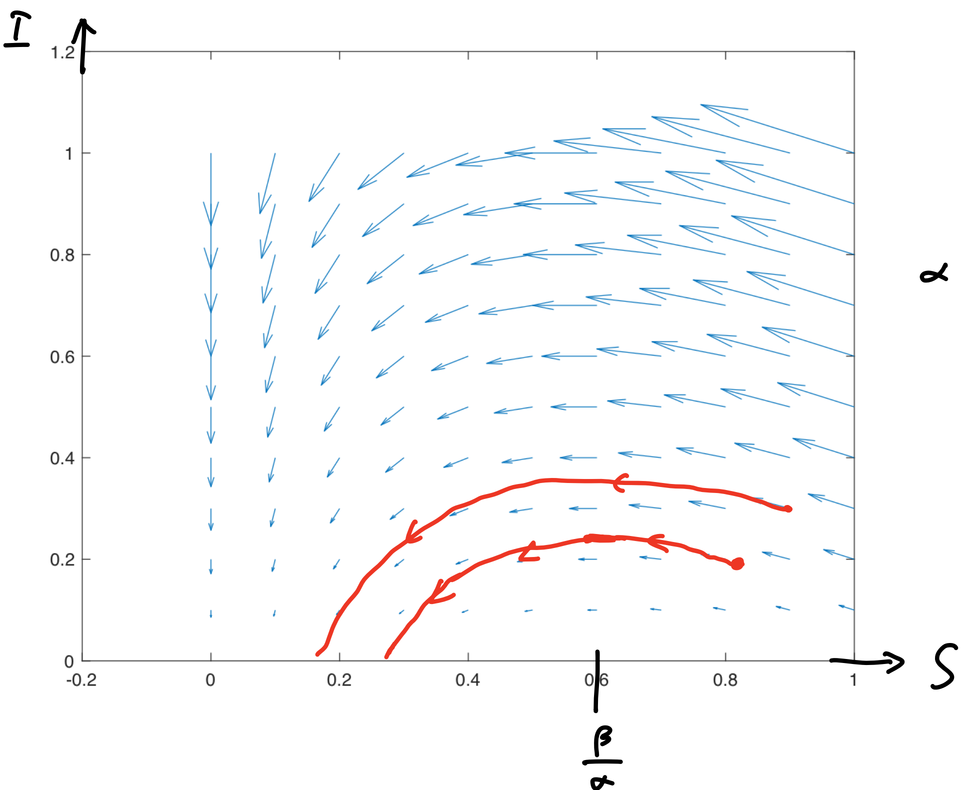


$$\frac{dI}{dt} = I(\alpha S - \beta). \quad \text{Suppose } I > 0$$

$$\Rightarrow \frac{dI}{dt} \begin{cases} = 0 & \text{if } S = \frac{\beta}{\alpha} \\ > 0 & \text{if } S > \frac{\beta}{\alpha} \\ < 0 & \text{if } S < \frac{\beta}{\alpha} \end{cases}$$

- If initially $I(t_0)$ is very small, then it can only start to increase (pandemic occurs) if $S(t_0) > \frac{\beta}{\alpha}$.

threshold value.



$\alpha = 1$, $\beta = 0.6$

$$\begin{cases} \dfrac{dS}{dt} = -\alpha S I \\ \dfrac{dI}{dt} = \alpha S I - \beta I = I(\alpha S - \beta) \end{cases}$$

- Solve for the sol'n curves.

$$\frac{dI}{dS} = \frac{I(\alpha S - \beta)}{-\alpha S I} = \frac{\alpha S - \beta}{-\alpha S} = -\frac{\alpha S - \beta}{\alpha S} = -\left(1 - \frac{\beta}{\alpha} \cdot \frac{1}{S}\right)$$

$$dI = -\left(1 - \frac{\beta}{\alpha} \cdot \frac{1}{S}\right) dS$$

$$I = -S + \frac{\beta}{\alpha} \cdot \ln|S| + C$$

$$I + S - \frac{\beta}{\alpha} \cdot \ln|S| = C$$

$\Rightarrow$ sol'n curves are level curves of $I + S - \frac{\beta}{\alpha} \cdot \ln|S|$

↓ below are not included in MIDTERM!

3.1 Linear systems

A <u>linear system of first order DEs</u> for $y_1(t), \dots, y_n(t)$ is

$$\begin{cases} \dfrac{dy_1}{dt} = a_{11}(t)\, y_1 + a_{12}(t)\, y_2 + \cdots + a_{1n}(t)\, y_n + b_1(t) \\ \vdots \\ \dfrac{dy_n}{dt} = a_{n1}(t)\, y_1 + a_{n2}(t)\, y_2 + \cdots + a_{nn}(t)\, y_n + b_n(t) \end{cases}$$

$$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

$$\vec{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$\frac{d\vec{Y}}{dt} = A(t)\vec{Y} + \vec{b}(t).$$

$$\frac{dy}{dt} = a(t)\, y + b(t)$$

- It is <u>homogeneous</u> if $\vec{b}(t) = 0$
- It is <u>homogeneous and has constant coefficients</u> if $\vec{b}(t) = 0$ and $A(t) = A$ is a constant matrix.

For example, damped harmonic oscillator

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

$$v = \frac{dy}{dt}$$

$$\begin{cases} \dfrac{dy}{dt} = v \\[2ex] \dfrac{dv}{dt} = -qy - pv \end{cases} \qquad \vec{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$$

$$\frac{d\vec{Y}}{dt} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}}_{A} \vec{Y} \qquad \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -qy - pv \end{pmatrix}$$

homogeneous w/ constant coefficients.

<u>Thm</u> (Linear principle)

Consider $\dfrac{d\vec{Y}}{dt} = A(t)\vec{Y} + \vec{b}(t)$ ------ (1)

$\dfrac{d\vec{Y}}{dt} = A(t)\vec{Y}$ ------ (2)

- If $\vec{Y}_1(t), \vec{Y}_2(t)$ are sol'ns to (2) and C is any constant, then $\vec{Y}_1 + \vec{Y}_2$ and $C\vec{Y}_1$ are also sol'ns to (2)

  (the sol'ns to (2) form a vector space)

- Let $\vec{Y}_p(t)$ be a particular sol'n to (1). Then any sol'n $\vec{Y}(t)$ to (1) has the form

$$\vec{Y}(t) = \vec{Y}_p(t) + \vec{Y}_h(t)$$

  where $\vec{Y}_h(t)$ is any sol'n to (2).

$\dfrac{dy}{dt} = a(t)y \rightsquigarrow y(t) = C$ ______ (sol'n space has dimension 1)

<u>Thm</u> If $\vec{Y}(t)$ has n components, then the vector space of sol'ns to (2) has dimension n. One can find a basis for it, $\vec{Y}_1(t), \ldots, \vec{Y}_n(t)$ such that general sol'n to (2) is

$$\vec{Y}(t) = C_1\vec{Y}_1(t) + \cdots + C_n\vec{Y}_n(t).$$