2.5 Euler's method for systems

$$\frac{d\vec{Y}}{dt} = \vec{F}(t, \vec{Y}) \quad , \quad \vec{Y}(t_0) = \vec{Y}_0$$

$$t_k = t_0 + k\Delta t$$

$$\vec{Y}_{k+1} = \vec{Y}_k + \Delta t \cdot \vec{F}(t_k, \vec{Y}_k)$$

$\vec{Y}_k$ is an approximation of $\vec{Y}(t_k)$

For a system of 2 DEs,

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\[2ex] \dfrac{dy}{dt} = g(t, x, y) \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\[2ex] y(t_0) = y_0 \end{cases}$$

Euler's method:

$$x_{k+1} = x_k + \Delta t \, f(t_k, x_k, y_k)$$

$$y_{k+1} = y_k + \Delta t \, g(t_k, x_k, y_k)$$

Ex Use Euler's method w/ $\Delta t = 0.5$ to solve

$$\begin{cases} \dfrac{dx}{dt} = y^2 - t \rightarrow f \\[2ex] \dfrac{dy}{dt} = x + y \rightarrow g \end{cases} \qquad \begin{cases} x(0) = 1 \\[2ex] y(0) = 0 \end{cases}$$

to approximate $x(1)$, $y(1)$

$t_0 = 0 \qquad t_1 = 0.5 \qquad t_2 = 1$

$$x_1 = x_0 + 0.5\, f(t_0, x_0, y_0) = 1 + 0.5\left(0^2 - 0\right) = 1$$

$$y_1 = y_0 + 0.5\, g(t_0, x_0, y_0) = 0 + 0.5\left(1 + 0\right) = 0.5$$

$$x_2 = x_1 + 0.5\, f(t_1, x_1, y_1) = 1 + 0.5\left(0.5^2 - 0.5\right) = \boxed{0.875}$$

$$y_2 = y_1 + 0.5\, g(t_1, x_1, y_1) = 0.5 + 0.5\left(1 + 0.5\right) = \boxed{1.25}$$

- To solve a high order DE by Euler's method, we need to first convert it into a system of first order DEs

Ex Use Euler's method w/ $\Delta t = 0.5$ to solve

$$\frac{d^2y}{dt^2} + 2y = t, \qquad y(1) = -2, \quad y'(1) = 3$$

to approximate $y(2)$.

Let $v(t) = \frac{dy}{dt}$. Then

$$\begin{cases} \dfrac{dy}{dt} = v \longrightarrow f \\[2ex] \dfrac{dv}{dt} = t - 2y \longrightarrow g \end{cases} \qquad \begin{cases} y(1) = -2 \\[2ex] v(1) = 3 \end{cases}$$

$t_0 = 1 \qquad t_1 = 1.5 \qquad t_2 = 2$

$$y_1 = y_0 + 0.5\, f(t_0, y_0, v_0) = -2 + 0.5\left(3\right) = -0.5$$

$$v_1 = v_0 + 0.5\, g(t_0, y_0, v_0) = 3 + 0.5\left(1 - 2 \times (-2)\right) = 5.5$$

$$y_2 = y_1 + 0.5\, f(t_1, y_1, v_1) = -0.5 + 0.5\left(5.5\right) = \boxed{2.25}$$

Solve decoupled systems (Sec. 2.4)

Consider a system of DEs for $x(t), y(t)$. If it has the form

$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ \dfrac{dy}{dt} = g(t, y) \end{cases}$$

then it is called completely decoupled. One can solve each DE seperately.

Ex Find general sol'n to

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\ \dfrac{dy}{dt} = \dfrac{2}{t} y + 3t \end{cases}$$

For x eq. :

$$\frac{dx}{x^2} = dt$$

$$-\frac{1}{x} = t + C_1$$

$$\frac{1}{x} = -(t + C_1)$$

$$x = -\frac{1}{t + C_1}$$

For y eq. :

$$\frac{dy}{dt} - \frac{2}{t} y = 3t$$

$$\mu(t) = e^{\int (-\frac{2}{t}) dt} = e^{-2 \ln t}$$

$$= \frac{1}{t^2}$$

$$\frac{d}{dt}\left(\frac{1}{t^2} y\right) = \frac{3}{t}$$

$$\frac{1}{t^2} y = 3 \ln|t| + C_2$$

$$y = 3t^2 \ln|t| + C_2 t^2$$

$$\Rightarrow \begin{cases} x = -\dfrac{1}{t + C_1} \\ y = 3t^2 \ln|t| + C_2 t^2 \end{cases}$$

A system of the form

$$\begin{cases} \frac{dx}{dt} = f(t, x, y) \\ \frac{dy}{dt} = g(t, y) \end{cases} \quad \text{or} \quad \begin{cases} \frac{dx}{dt} = f(t, x) \\ \frac{dy}{dt} = g(t, x, y) \end{cases}$$

is called <u>partially decoupled</u>. One can solve one DE first, and substitute into another.

<u>Ex</u> Solve

$$\begin{cases} \frac{dx}{dt} = -x + y^2 \\ \frac{dy}{dt} = 2y \end{cases} \qquad \begin{cases} x(0) = 2 \\ y(0) = -1 \end{cases}$$

Solve y first:

$$\frac{dy}{y} = 2\,dt$$

$$\ln|y| = 2t + C_1$$

$$\ln(-y) = 2t + C_1 \quad (\text{since } y(0) = -1 < 0)$$

initial condition $\Rightarrow$

$$\ln(1) = 2 \cdot 0 + C_1 \quad \Rightarrow C_1 = 0$$

$$\ln(-y) = 2t$$

$$-y = e^{2t}$$

$$y = -e^{2t}$$

Then solve x:

$$\frac{dx}{dt} = -x + e^{4t}$$

$$\frac{dx}{dt} + x = e^{4t}$$

$$\mu(t) = e^{\int 1\,dt} = e^t$$

$$\frac{d}{dt}(e^t x) = e^{5t}$$

$$e^t x = \frac{1}{5} e^{5t} + C_2$$

initial condition $\Rightarrow$

$$e^0 \cdot 2 = \frac{1}{5} e^{5 \cdot 0} + C_2$$

$$2 = \frac{1}{5} + C_2$$

$$C_2 = \frac{9}{5}$$

$$e^t x = \frac{1}{5} e^{5t} + \frac{9}{5}$$

$$x = \frac{1}{5}e^{4t} + \frac{9}{5}e^{-t}$$

$$\Rightarrow \begin{cases} x = \frac{1}{5}e^{4t} + \frac{9}{5}e^{-t} \\ y = -e^{2t} \end{cases}$$

Ex Solve

$$\begin{cases} \frac{dx}{dt} = 3z \\ \frac{dy}{dt} = \frac{x+z}{2y} \\ \frac{dz}{dt} = t^2 \end{cases} \qquad \begin{cases} x(0) = 2 \\ y(0) = 1 \\ z(0) = 0 \end{cases}$$

Solve z first :

$$dz = t^2\,dt$$

$$z = \frac{1}{3}t^3 + C_1$$

initial condition $\Rightarrow$

$$0 = \frac{1}{3}\cdot 0^3 + C_1 \Rightarrow C_1 = 0$$

$$z = \frac{1}{3}t^3$$

Then solve x :

$$dx = t^3\,dt$$

$$x = \frac{1}{4}t^4 + C_2$$

initial condition $\Rightarrow$

$$2 = \frac{1}{4}\cdot 0^4 + C_2 \Rightarrow C_2 = 2$$

$$x = \frac{1}{4}t^4 + 2$$

Finally solve y :

$$\Rightarrow \begin{cases} \frac{dx}{dt} = t^3 \\ \frac{dy}{dt} = \frac{x + \frac{1}{3}t^3}{2y} \end{cases}$$

$$\Rightarrow \frac{dy}{dt} = \frac{\frac{1}{4}t^4 + 2 + \frac{1}{3}t^3}{2y}$$

$$2y\,dy = \left(\frac{1}{4}t^4 + 2 + \frac{1}{3}t^3\right)dt$$

$$y^2 = \frac{1}{20}t^5 + 2t + \frac{1}{12}t^4 + C_3$$

initial condition $\Rightarrow$

$$1^2 = \frac{1}{20}\cdot 0^5 + 2\cdot 0 + \frac{1}{12}\cdot 0^4 + C_3 \quad \Rightarrow \quad C_3 = 1$$

$$y^2 = \frac{1}{20}t^5 + 2t + \frac{1}{12}t^4 + 1$$

$$y = \sqrt{\frac{1}{20}t^5 + 2t + \frac{1}{12}t^4 + 1}$$

$$\Rightarrow \begin{cases} x = \frac{1}{4}t^4 + 2 \\ y = \sqrt{\frac{1}{20}t^5 + 2t + \frac{1}{12}t^4 + 1} \\ z = \frac{1}{3}t^3 \end{cases}$$

$$\frac{d^2y}{dt^2} = -y \qquad v = \frac{dy}{dt}$$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -y \end{cases}$$