2.3 (Continued)

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

Try $y(t) = e^{st}$ $\Rightarrow$ $s^2 + ps + q = 0$

$$s = \frac{-p \pm \sqrt{p^2-4q}}{2}$$

Case 3 : $p^2 - 4q = 0$

$\Rightarrow$ $s = -\frac{p}{2}$ $\qquad$ $y_1(t) = e^{-\frac{p}{2}t}$ is a particular sol'n.

Fact : $y_2(t) = t e^{-\frac{p}{2}t}$ is also a particular sol'n.

$$\frac{dy_2}{dt} = e^{-\frac{p}{2}t} + t\cdot(-\tfrac{p}{2})e^{-\frac{p}{2}t}$$

$$= e^{-\frac{p}{2}t}\left(1 - \tfrac{p}{2}t\right)$$

$$\frac{d^2y_2}{dt^2} = (-\tfrac{p}{2})e^{-\frac{p}{2}t}\left(1-\tfrac{p}{2}t\right) + e^{-\frac{p}{2}t}\cdot(-\tfrac{p}{2})$$

$$= e^{-\frac{p}{2}t}\left(-\tfrac{p}{2} + \tfrac{p^2}{4}t - \tfrac{p}{2}\right)$$

$$= e^{-\frac{p}{2}t}\left(-p + \tfrac{p^2}{4}t\right)$$

$q = \frac{p^2}{4}$

$$\frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2 = e^{-\frac{p}{2}t}\left(-p + \tfrac{p^2}{4}t + p\left(1-\tfrac{p}{2}t\right) + qt\right)$$

$$= e^{-\frac{p}{2}t}\left(\cancel{-p} + \tfrac{p^2}{4}t \cancel{+p} - \tfrac{p^2}{2}t + \tfrac{p^2}{4}t\right) = 0$$

Summarize for $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$

Solve $s^2 + ps + q = 0$ "characteristic equation"

$$s = \frac{-p \pm \sqrt{p^2-4q}}{2}$$

Case 1 : $p^2 - 4q > 0 \Rightarrow$ get particular sol'ns

$y_1(t) = e^{s_1 t}$, $y_2(t) = e^{s_2 t}$

Case 2 : $p^2 - 4q < 0 \Rightarrow$ get particular sol'ns

$$y_1(t) = e^{-\frac{p}{2}t}\cos\left(\frac{\sqrt{4q-p^2}}{2}t\right)$$

$$y_2(t) = e^{-\frac{p}{2}t}\sin\left(\frac{\sqrt{4q-p^2}}{2}t\right)$$

Case 3 : $p^2 - 4q = 0 \Rightarrow$ get particular sol'ns

$y_1(t) = e^{-\frac{p}{2}t}$, $y_2(t) = te^{-\frac{p}{2}t}$

To solve an initial value problem, try

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

and determine C_1, C_2 by initial condition.

Ex Solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0, \quad y(0) = 1, \; y'(0) = 2$$

$$s^2 + 2s + 2 = 0$$

$$s = \frac{-2 \pm \sqrt{2^2 - 4\cdot1\cdot2}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = \underbrace{-1}_{\text{coeff}=-1} \pm \underbrace{i}_{\text{coeff}=1}$$

$\Rightarrow y_1(t) = e^{-t}\cos t$

$y_2(t) = e^{-t}\sin t$

$$y(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$$

$$y'(t) = C_1(-e^{-t}\cos t - e^{-t}\sin t) + C_2(-e^{-t}\sin t + e^{-t}\cos t)$$

$$y(0) = C_1 \qquad y'(0) = -C_1 + C_2$$

initial condition $\Rightarrow \begin{cases} C_1 = 1 \\ -C_1 + C_2 = 2 \end{cases} \Rightarrow C_2 = 3$

$$\Rightarrow y(t) = e^{-t}\cos t + 3e^{-t}\sin t .$$

<u>Ex</u> Solve

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1,\ y'(0) = 2$$

$$s^2 + 4s + 4 = 0 \qquad (s+2)^2 = 0 \qquad s = -2$$

$$y_1(t) = e^{-2t} \qquad y_2(t) = te^{-2t}$$

$$y(t) = C_1 e^{-2t} + C_2 te^{-2t}$$

$$y'(t) = -2C_1 e^{-2t} + C_2(e^{-2t} - 2te^{-2t})$$

$$y(0) = C_1 \qquad y'(0) = -2C_1 + C_2$$

initial condition $\Rightarrow \begin{cases} C_1 = 1 \\ -2C_1 + C_2 = 2 \end{cases} \Rightarrow C_2 = 4$

$$\Rightarrow y(t) = e^{-2t} + 4te^{-2t}$$

2.6 Existence and uniqueness of systems

General system of first order DEs for $y_1(t), \cdots, y_n(t)$

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, \cdots, y_n) \\ \vdots \\ \dfrac{dy_n}{dt} = f_n(t, y_1, \cdots, y_n) \end{cases}$$

initial condition

$$\begin{cases} y_1(t_0) = y_{1,0} \\ \vdots \\ y_n(t_0) = y_{n,0} \end{cases}$$

$$\vec{Y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \qquad \vec{F}(t, \vec{Y}) = \begin{pmatrix} f_1(t, y_1, \cdots, y_n) \\ \vdots \\ f_n(t, y_1, \cdots, y_n) \end{pmatrix}$$

$$\vec{Y}_0 = \begin{pmatrix} y_{1,0} \\ \vdots \\ y_{n,0} \end{pmatrix}$$

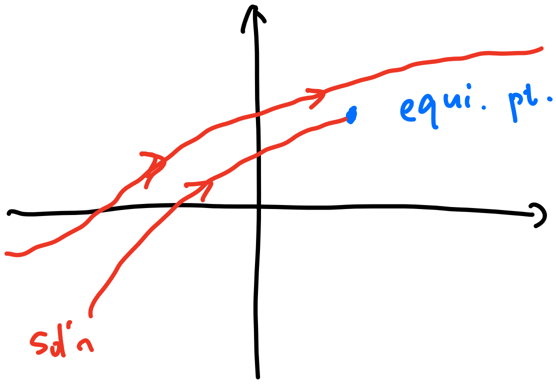
$$\frac{d\vec{Y}}{dt} = \vec{F}(t, \vec{Y}), \quad \text{initial condition: } \vec{Y}(t_0) = \vec{Y}_0 \qquad (1)$$

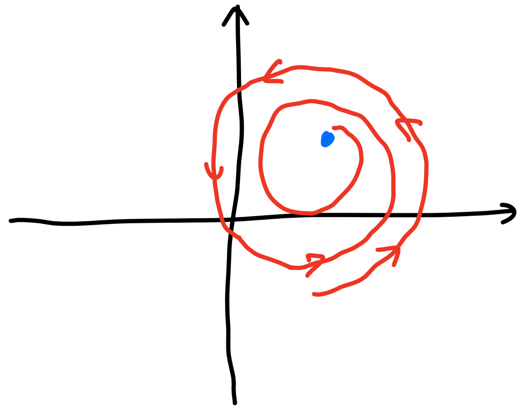
Thm If $\vec{F}(t, \vec{Y})$ and $\dfrac{\partial \vec{F}}{\partial \vec{Y}}$ are continuous, then there exists

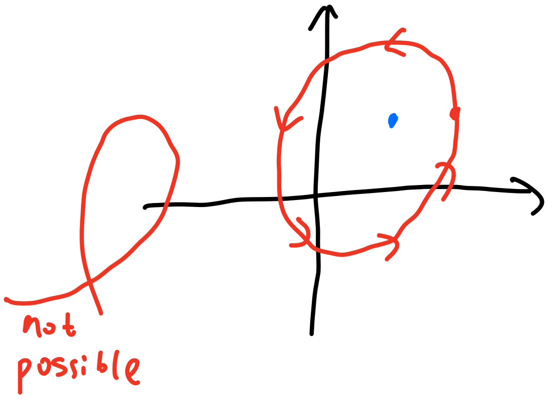
$\hookrightarrow \dfrac{\partial f_i}{\partial y_j}$ where $i, j = 1, \cdots, n$

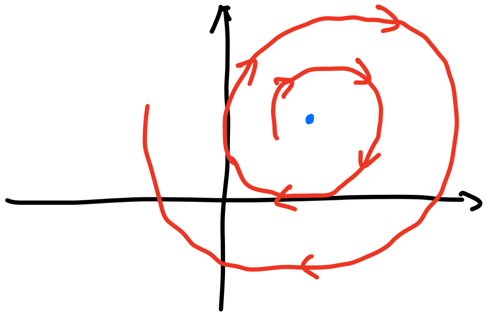
$\varepsilon > 0$ such that the initial value problem (1) has a unique sol'n $\vec{Y}(t)$ defined on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

- For autonomous systems, sol'n curves cannot intersect (w/ each other or w/ itself).









- For a p-th order DE for $y(t)$

$$\frac{d^p y}{dt^p} = f\left(t, y, \frac{dy}{dt}, \cdots, \frac{d^{p-1}y}{dt^{p-1}}\right)$$

Let

$$y_1(t) = y(t)$$
$$y_2(t) = \frac{dy}{dt}$$
$$y_3(t) = \frac{d^2y}{dt^2}$$
$$\vdots$$
$$y_{p-1}(t) = \frac{d^{p-2}y}{dt^{p-2}}$$
$$y_p(t) = \frac{d^{p-1}y}{dt^{p-1}}$$

$$\Rightarrow \begin{cases} \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = y_3 \\ \vdots \\ \frac{dy_{p-1}}{dt} = y_p \\ \frac{dy_p}{dt} = f(t, y_1, y_2, \cdots, y_p) \end{cases}$$

Initial values $\begin{cases} y_1(t_0) = y_{1,0} \\ y_2(t_0) = y_{2,0} \\ \vdots \\ y_p(t_0) = y_{p,0} \end{cases}$

$\Rightarrow$ for p-th order DE, one needs initial values $\begin{cases} y(t_0) = y_{1,0} \\ \frac{dy}{dt}(t_0) = y_{2,0} \\ \vdots \\ \frac{d^{p-1}y}{dt^{p-1}}(t_0) = y_{p,0} \end{cases}$.