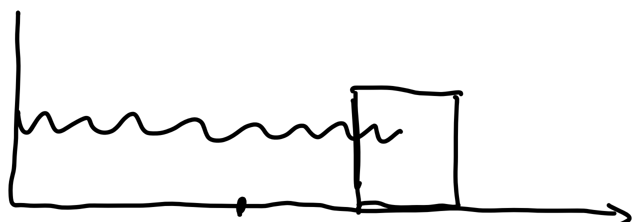
2.3 The damped harmonic oscillator



$y(t)$: position of the block

m : mass - - - - - -

k : spring constant

$$m\frac{d^2y}{dt^2} = \underbrace{-ky}_{\text{force from spring}} \qquad \text{(harmonic oscillator)}$$

If one considers friction proportional to velocity (for air friction)

$$m\frac{d^2y}{dt^2} = \underbrace{-ky}_{\text{force from spring}} \underbrace{-b\frac{dy}{dt}}_{\text{friction}}$$

b : damping coefficient

(damped harmonic oscillator)

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$p = \frac{b}{m} > 0, \quad q = \frac{k}{m} > 0$$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

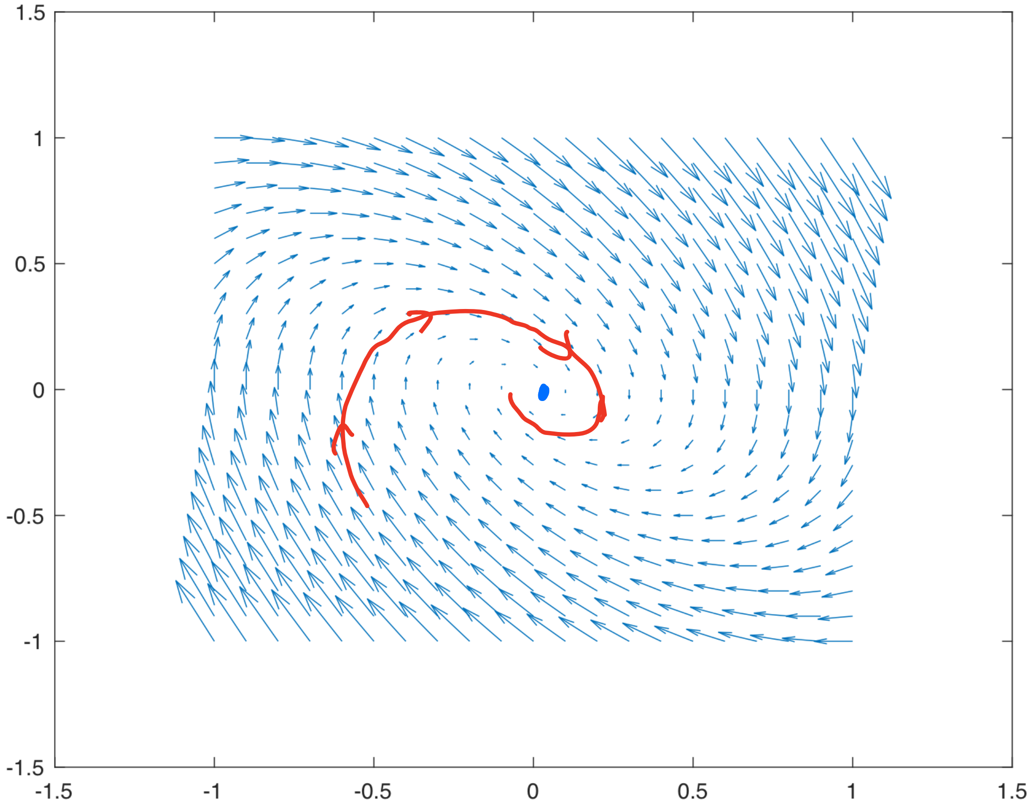
Linear 2nd order DE. (homogeneous)

$$v(t) = \frac{dy}{dt}$$

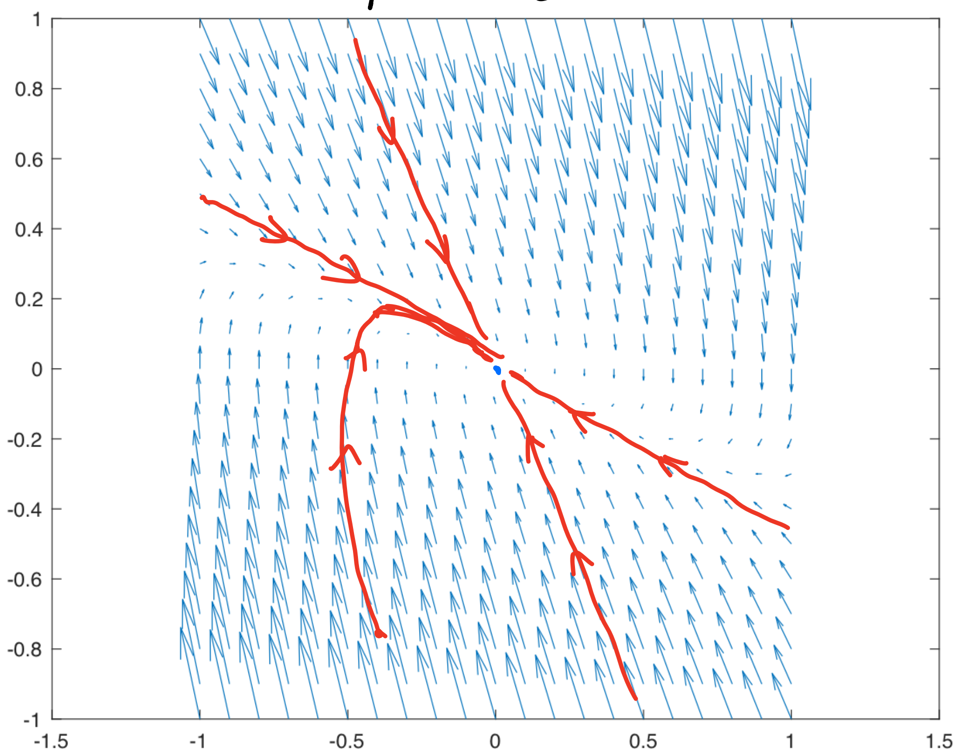
$$\begin{cases} \dfrac{dy}{dt} = v \\[2ex] \dfrac{dv}{dt} = -pv - qy \end{cases}$$

equilibrium pt : $(y, v) = (0, 0)$

Case: $p=1$, $q=1$



$p=3$, $q=1$



$$s^2 + 3s + 1 = 0$$

$$s = \frac{-3 \pm \sqrt{3^2 - 4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$\frac{dy}{dt} = ay \qquad y = e^{at}$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

Try sol'n $\quad y(t) = e^{st}$ (where s is a parameter to be determined)

$$\frac{dy}{dt} = se^{st}$$

$$\frac{d^2y}{dt^2} = s^2 e^{st}$$

$$\Rightarrow \quad s^2 e^{st} + ps\,e^{st} + q\,e^{st} = 0$$

$$e^{st}(s^2 + ps + q) = 0$$

$\Rightarrow$ Need $\quad s^2 + ps + q = 0$

$$s = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Case 1: $\quad p^2 - 4q > 0$. We get two real roots s_1, s_2

$\Rightarrow$ 2 particular sol'ns

$$y_1(t) = e^{s_1 t}, \qquad y_2(t) = e^{s_2 t}$$

$y_1(t) = e^{s_1 t}$

$v_1(t) = s_1 e^{s_1 t}$

$\Rightarrow$ Sol'n curve $\quad v = s_1 y$

Case 2: $\quad p^2 - 4q < 0$. We get two complex roots

$$s_{1,2} = -\frac{p}{2} \pm \frac{\sqrt{4q - p^2}}{2}\,i$$

$\Rightarrow$ 2 complex particular sol'ns

$$y_1(t) = e^{\left(-\frac{p}{2} + \frac{\sqrt{4q-p^2}}{2} i\right)t}$$
$$= e^{-\frac{p}{2}t} e^{\frac{\sqrt{4q-p^2}}{2} t i}$$
$$= e^{-\frac{p}{2}t}\left(\cos\left(\frac{\sqrt{4q-p^2}}{2}t\right) + i \sin\left(\frac{\sqrt{4q-p^2}}{2}t\right)\right)$$

DE is linear, homogeneous.

$\Rightarrow$ linear combinations of sol'ns are still sol'ns

$e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$y_2(t) = e^{\left(-\frac{p}{2} - \frac{\sqrt{4q-p^2}}{2} i\right)t}$$
$$= e^{-\frac{p}{2}t} e^{-\frac{\sqrt{4q-p^2}}{2} t i}$$
$$= e^{-\frac{p}{2}t}\left(\cos\left(\frac{\sqrt{4q-p^2}}{2}t\right) - i \sin\left(\frac{\sqrt{4q-p^2}}{2}t\right)\right)$$

$\Rightarrow$ real sol'n $y_3(t) = \frac{1}{2}(y_1(t) + y_2(t))$

$$= e^{-\frac{p}{2}t} \cos\left(\frac{\sqrt{4q-p^2}}{2} t\right)$$

$$y_4(t) = \frac{1}{2i}(y_1(t) - y_2(t))$$
$$= e^{-\frac{p}{2}t} \sin\left(\frac{\sqrt{4q-p^2}}{2} t\right)$$

Case 3 : $p^2 - 4q = 0$ (to be done)

- As long as we find two sol'ns $y_1(t)$, $y_2(t)$ (which are not multiples of each other), then we can solve any initial value problem by setting $y(t) = C_1 y_1(t) + C_2 y_2(t)$.

Ex Solve

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 2$$

① $s^2 + 5s + 4 = 0$

$(s+1)(s+4) = 0$

$s_1 = -1, \quad s_2 = -4$

$\Rightarrow$ particular sol'ns to DE : $y_1 = e^{-t}, \quad y_2 = e^{-4t}$

② $y(t) = C_1 e^{-t} + C_2 e^{-4t}$

$y'(t) = -C_1 e^{-t} - 4C_2 e^{-4t}$

Initial condition $\Rightarrow$ $\begin{cases} C_1 + C_2 = 1 \\ -C_1 - 4C_2 = 2 \end{cases}$

$-3C_2 = 3 \qquad C_2 = -1 \qquad C_1 = 2$

$\Rightarrow y(t) = 2e^{-t} - e^{-4t}$