2.1, 2.2 (Continued)

Consider an autonomous system of 2 DEs :

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases}$$

$$\frac{dy}{dt} = f(y)$$

- equilibrium pts, phase line
- separable eq.

<u>Equilibrium pts</u> : a choice of (x,y) which makes $f(x,y)=0$, $g(x,y)=0$.

This gives a constant sol'n.

<u>Ex</u> Find equilibrium pts of

$$\begin{cases} \dfrac{dR}{dt} = R - 2RF \\[2ex] \dfrac{dF}{dt} = -2F + 3RF \end{cases}$$

Let $\begin{cases} R - 2RF = 0 \\ -2F + 3RF = 0 \end{cases} \Rightarrow \begin{cases} R(1-2F) = 0 \\ F(-2+3R) = 0 \end{cases}$

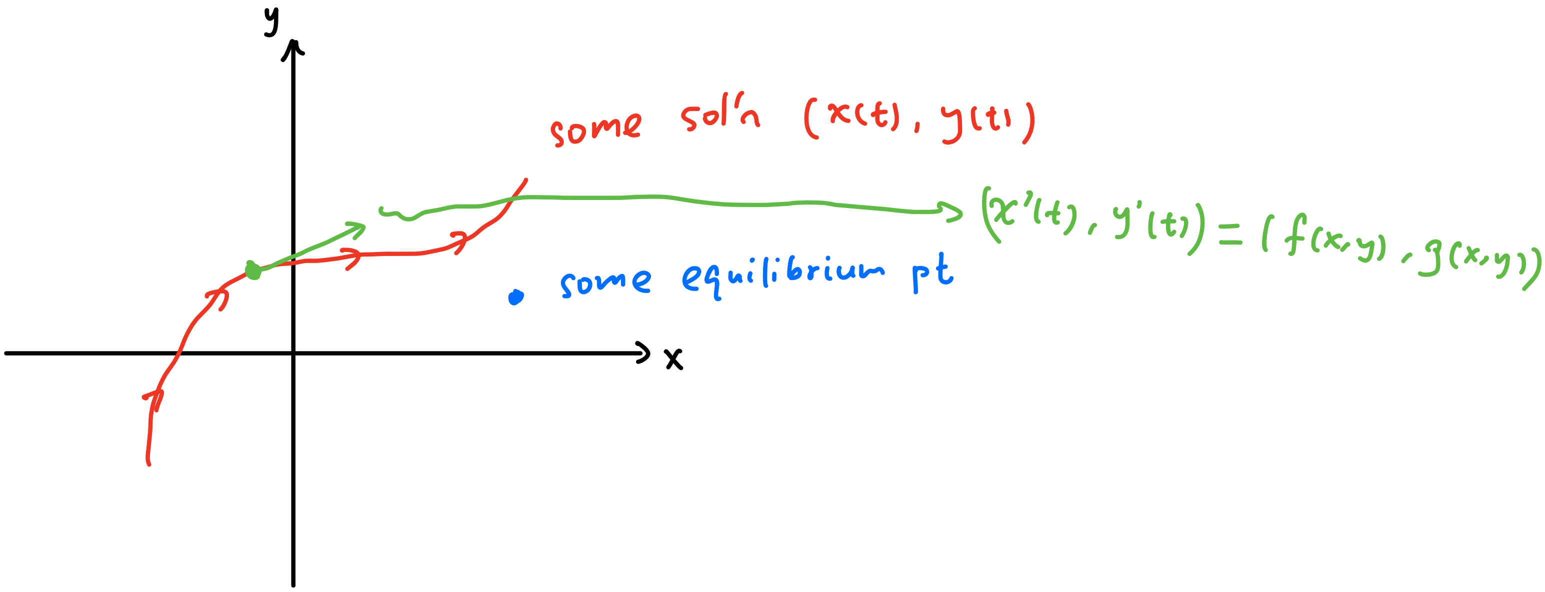
$\Rightarrow \begin{cases} R = 0 \text{ or } 1 - 2F = 0 \\ F = 0 \text{ or } -2 + 3R = 0 \end{cases}$

If $R=0$, then $-2+3R \neq 0 \Rightarrow F = 0$

Otherwise, $1-2F=0 \Rightarrow F = \frac{1}{2}$, then $-2+3R=0 \Rightarrow R = \frac{2}{3}$

$\Rightarrow$ Equilibrium pts $(R,F) = (0,0)$ or $(\frac{2}{3}, \frac{1}{2})$.

Phase portrait : view each sol'n $(x(t), y(t))$ as a moving point in the (x,y)-plane

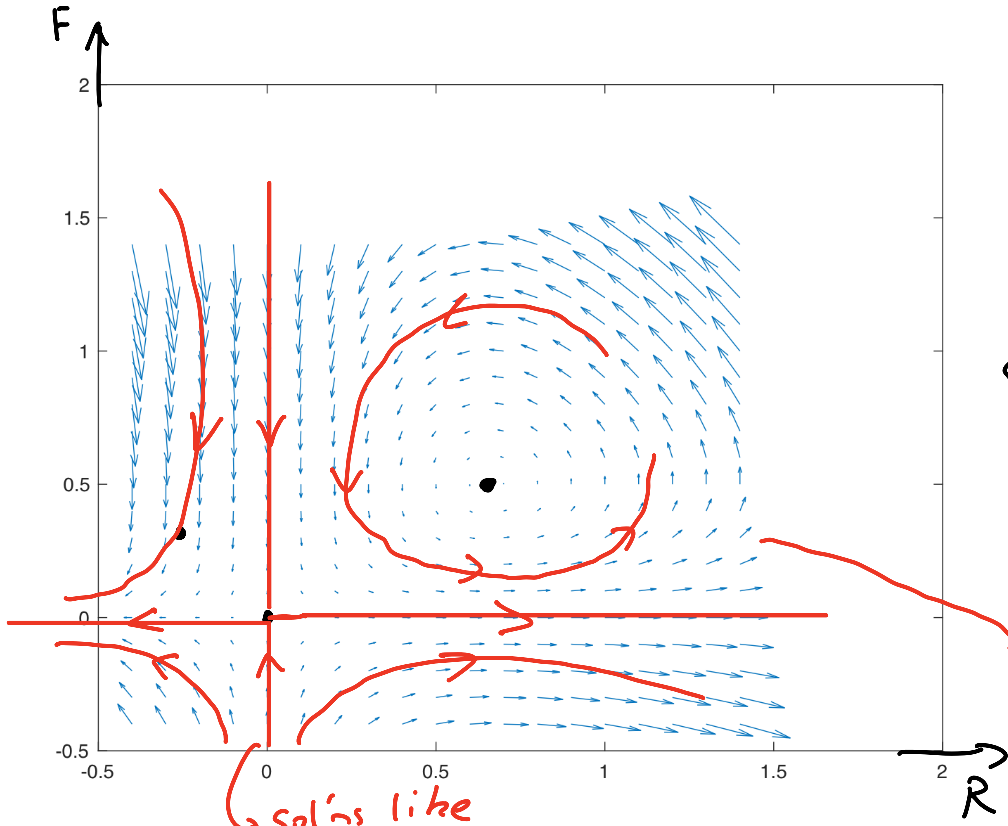


To get some sol'ns for the system:

- Observe and find special ones.
- Use vector field: any sol'n passing (x,y) has tangent vector $(f(x,y), g(x,y))$ $\Rightarrow$ if we draw a vector $(f(x,y), g(x,y))$ at each (x,y), then we can follow the vectors and sketch the sol'ns.
- Get equation of sol'n curves by

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \rightsquigarrow \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

(view $y = y(x)$)



$$\begin{cases} \dfrac{dR}{dt} = R - 2RF \\[2ex] \dfrac{dF}{dt} = -2F + 3RF \end{cases}$$

sol'ns like
$$\begin{cases} R(t) = Ce^{t} \\ F(t) = 0 \end{cases}$$

sol'ns like
$$\begin{cases} R(t) = 0 \\ F(t) = Ce^{-2t} \end{cases}$$

Ex Find equations of the sol'n curves for

$$\begin{cases} \dfrac{dR}{dt} = R - 2RF \\[2ex] \dfrac{dF}{dt} = -2F + 3RF \end{cases}$$

$$\frac{dF}{dR} = \frac{-2F + 3RF}{R - 2RF} \qquad \text{(viewing } F = F(R)\text{)}$$

$$\frac{dF}{dR} = \frac{F(-2 + 3R)}{R(1 - 2F)}$$

$$\frac{1-2F}{F}\, dF = \frac{-2+3R}{R}\, dR \qquad \frac{1-2F}{F} = \frac{1}{F} - 2$$

$$\int \frac{1-2F}{F}\, dF = \int \frac{-2+3R}{R}\, dR$$

$$\ln|F| - 2F = -2\ln|R| + 3R + C$$

Sol'n curves are level curves of

$$\ln|F| - 2F + 2\ln|R| - 3R$$

(suggesting that certain sol'n curves are closed curves).

- 2nd order DE can be converted to a system of 2 1st order DEs.

<u>Ex</u> The harmonic oscillator

$$\frac{d^2y}{dt^2} = -y$$

Let $v(t) = \dfrac{dy}{dt}$

$$\begin{cases} \dfrac{dy}{dt} = v \\[2ex] \dfrac{dv}{dt} = -y \end{cases}$$

Generally, a p-th order DE in $y(t)$ can be converted to a system of p 1st DEs in $\left(y, \dfrac{dy}{dt}, \ldots, \dfrac{d^{p-1}y}{dt^{p-1}}\right)$

vector field $(v, -y)$

Equilibrium pts : $(y, v) = (0, 0)$

To get sol'n curves :

$$\frac{dv}{dy} = \frac{-y}{v}$$

$$v\,dv = -y\,dy$$

$$\frac{1}{2}v^2 = -\frac{1}{2}y^2 + C$$

$$\frac{1}{2}v^2 + \frac{1}{2}y^2 = C$$

$$v^2 + y^2 = 2C$$