1.5 (continued)

$$\frac{dy}{dt} = f(t,y)$$

$$\frac{\partial f}{\partial y}$$

Ex $\dfrac{dy}{dt} = y^{\frac{1}{3}}$, $y(0) = 0$

$f(t,y) = y^{\frac{1}{3}}$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-\frac{2}{3}}$ is not continuous at $y=0$.

$$\frac{dy}{y^{\frac{1}{3}}} = dt$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + C \qquad \frac{3}{2}\cdot 0^{\frac{2}{3}} = 0 + C \qquad C = 0$$

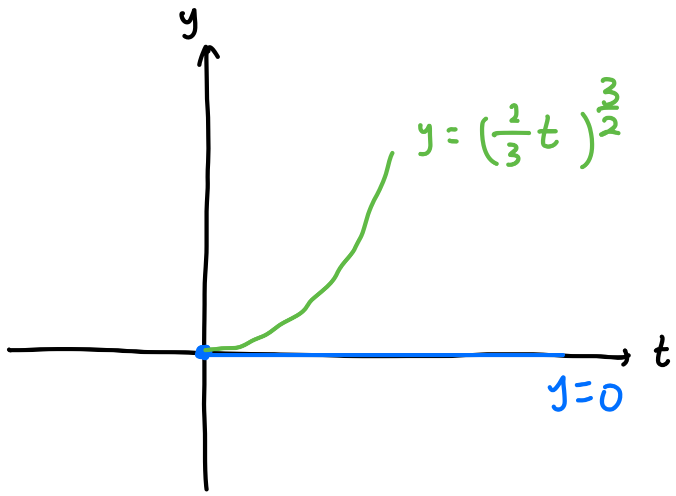
$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$y^{\frac{2}{3}} = \frac{2}{3} t$$

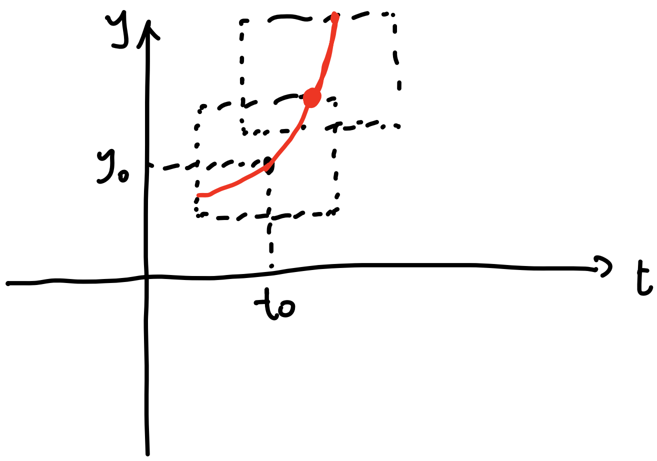
$$y = \left(\frac{2}{3} t\right)^{\frac{3}{2}}$$

Notice: $y=0$ is also a sol'n

Failure of uniqueness.



Finite time blow up



Even if $f(t,y)$ is nice enough, not every sol'n can be extended to all time.

Sol'n can blow up to $\infty / -\infty$ within finite time.

Ex $\frac{dy}{dt} = y^2$, $y(0) = 1$

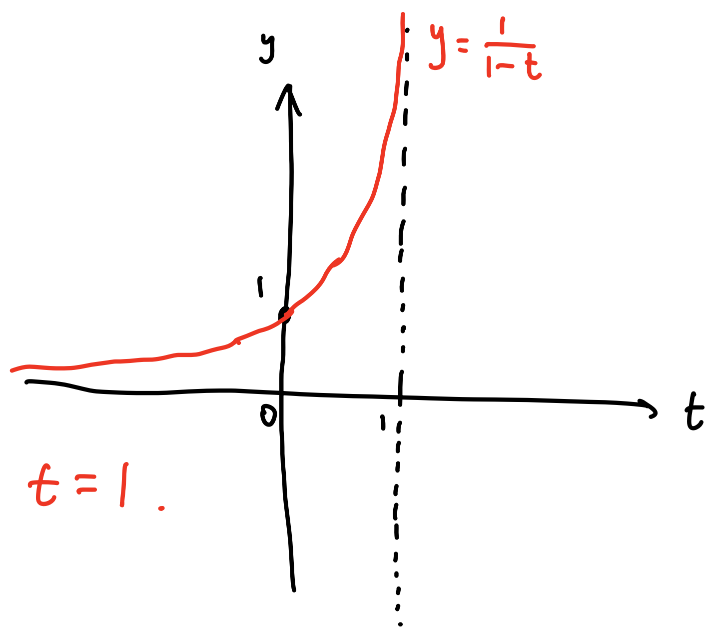
$$\frac{dy}{y^2} = dt$$

$$-\frac{1}{y} = t + C \qquad -\frac{1}{1} = 0 + C \qquad C = -1$$

$$-\frac{1}{y} = t - 1$$

$$\frac{1}{y} = 1 - t$$

$$y = \frac{1}{1-t}$$



finite time blow up to ∞ at $t = 1$.

(domain of definition $(-\infty, 1)$)

Domain of definition of a particular sol'n :

the maximal interval in t for which some sol'n can be extended as a differentiable function

Ex $\frac{dy}{dt} = y^2+1, \; y(0) = 1$ determine domain of def. of the sol'n.

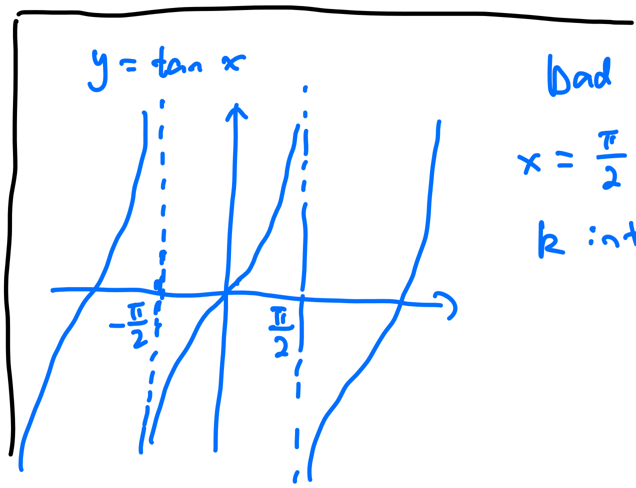
$$\frac{dy}{y^2+1} = dt$$

$\tan^{-1}(y) = t + C$ $\quad\quad \tan^{-1}(1) = 0 + C$

$$C = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\tan^{-1}(y) = t + \frac{\pi}{4}$$

$$y = \tan\left(t + \frac{\pi}{4}\right)$$



bad pts: $x = \frac{\pi}{2} + k\pi$

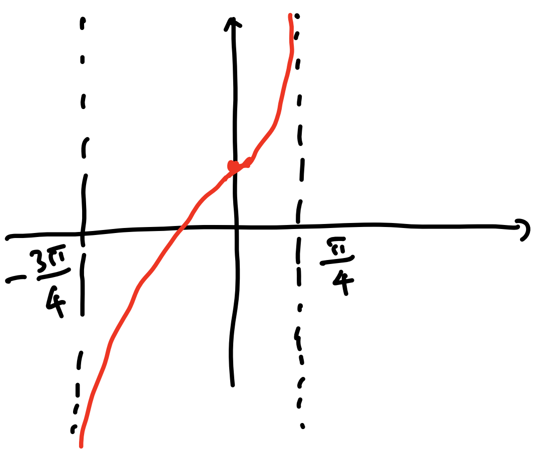
k integer

right end pt: $t + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

left end pt: $t + \frac{\pi}{4} = -\frac{\pi}{2} \Rightarrow t = -\frac{3\pi}{4}$

$\Rightarrow$ domain of def. is

$$\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$$



Ex $\frac{dy}{dt} = t^3(1+y^2), \; y(0) = 1$ determine domain of def. of the sol'n.

$$\frac{dy}{1+y^2} = t^3\, dt$$

$\tan^{-1}(y) = \frac{1}{4}t^4 + C$ $\quad\quad \tan^{-1}(1) = \frac{1}{4}\cdot 0^4 + C$ $\quad\quad C = \frac{\pi}{4}$

$$\tan^{-1} y = \frac{1}{4}t^4 + \frac{\pi}{4}$$

$$y = \tan\left(\frac{1}{4}t^4 + \frac{\pi}{4}\right)$$

right end pt : $\frac{1}{4}t^4 + \frac{\pi}{4} = \frac{\pi}{2}$

$$\frac{1}{4}t^4 = \frac{\pi}{4}$$

$$t^4 = \pi$$

$$t = \pm \pi^{1/4} \qquad \text{take } t = \pi^{1/4}$$

left end pt : $\frac{1}{4}t^4 + \frac{\pi}{4} = \frac{\pi}{2}$ take $t = -\pi^{1/4}$

$\Rightarrow \quad (-\pi^{1/4}, \pi^{1/4})$.

- Linear DE always has domain of def. $(-\infty, \infty)$ for any sol'n.

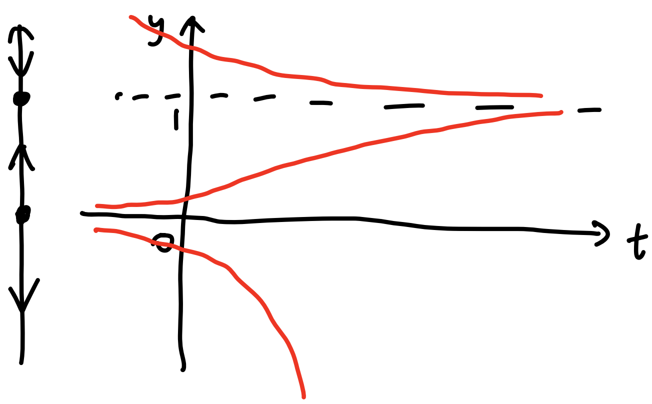
$$\frac{dy}{dt} = a(t)y + b(t).$$

- Blow up may happen if $|f(t,y)|$ can grow faster than linear as $|y|$ gets large.

$\frac{dy}{dt} = y^{1.01} \qquad y(0) = 1$

- For one DE, it may happen that some sol'ns are globally defined (domain of def. $(-\infty, \infty)$) but other sol'ns blow up.

<u>Ex</u> $\frac{dy}{dt} = y(1-y)$



2.1, 2.2 Modeling via systems, geometry of systems

$y_1(t), \cdots, y_n(t)$

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, \cdots, y_n) \\ \cdots \\ \dfrac{dy_n}{dt} = f_n(t, y_1, \cdots, y_n) \end{cases}$$

System of DEs w/ 2 unknown funcs $x(t), y(t)$

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y) \\ \dfrac{dy}{dt} = g(t, x, y) \end{cases}$$

Autonomous system of DEs w/ 2 unknown funcs $x(t), y(t)$

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases}$$

<u>Ex</u> Predator-prey model

Say, rabbits : $R(t)$

foxes : $F(t)$

rabbit has exp. growth w/o fox

rabbit eaten by fox

$$\begin{cases} \dfrac{dR}{dt} = \alpha R - \beta R F \end{cases}$$

$\alpha, \beta, \gamma, \delta > 0$.

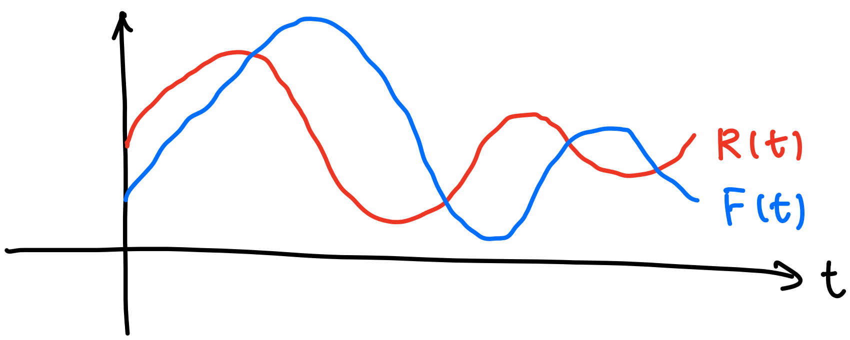
$$\frac{dF}{dt} = -\gamma F + \delta R F$$

fox has exp. decay w/o rabbit.

eating rabbit helps fox grow.

To illustrate a sol'n to a system of DEs,

- Graphs of sol'ns.



- Autonomous systems: <u>phase portrait</u> (next time).