1.7 Bifurcations

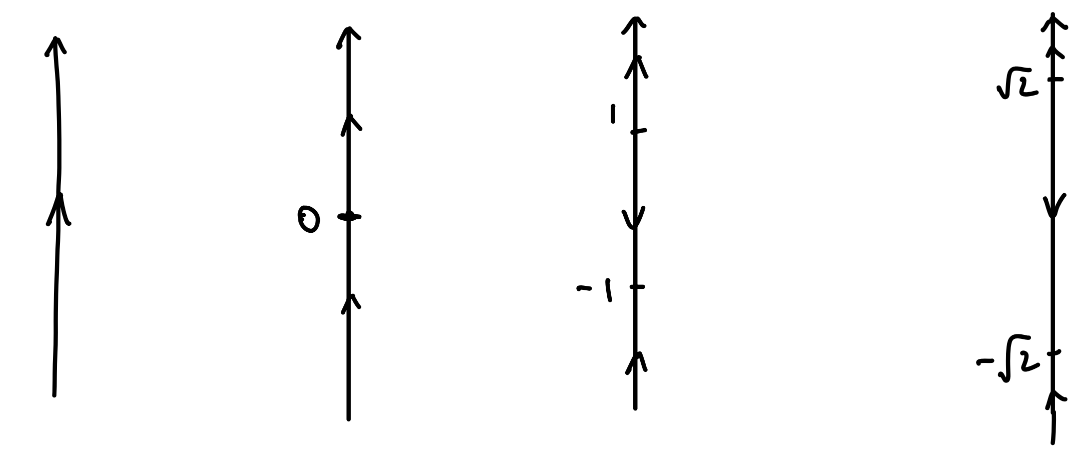
Consider an autonomous DE w/ a parameter μ

$$\frac{dy}{dt} = f_\mu(y) \qquad \left(\text{for example, } \frac{dy}{dt} = y^2 - \mu \right)$$

For different values of μ, the qualitative behavior of its phase line may be different.

bifurcation value

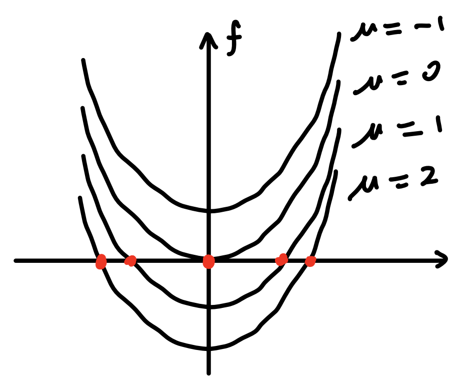
$\mu = -1$ $\left(\frac{dy}{dt} = y^2 + 1\right)$ $\qquad$ $\boxed{\mu = 0}$ $\left(\frac{dy}{dt} = y^2\right)$ $\qquad$ $\mu = 1$ $\left(\frac{dy}{dt} = y^2 - 1\right)$ $\qquad$ $\mu = 2$ $\left(\frac{dy}{dt} = y^2 - 2\right)$



A value of μ around which the qualitative behavior of the phase line changes is called a <u>bifurcation value</u>

Fact: for a bifurcation value μ, the corresponding $f_\mu(y)$ must have a multiple root (that is, some y_* satisfying $f_\mu(y_*) = f'_\mu(y_*) = 0$)

$f_\mu(y) = y^2 - \mu$



Suppose $f_\mu(y)$ is a quadratic function in y.

Then $f_\mu(y)$ has a multiple root if and only if

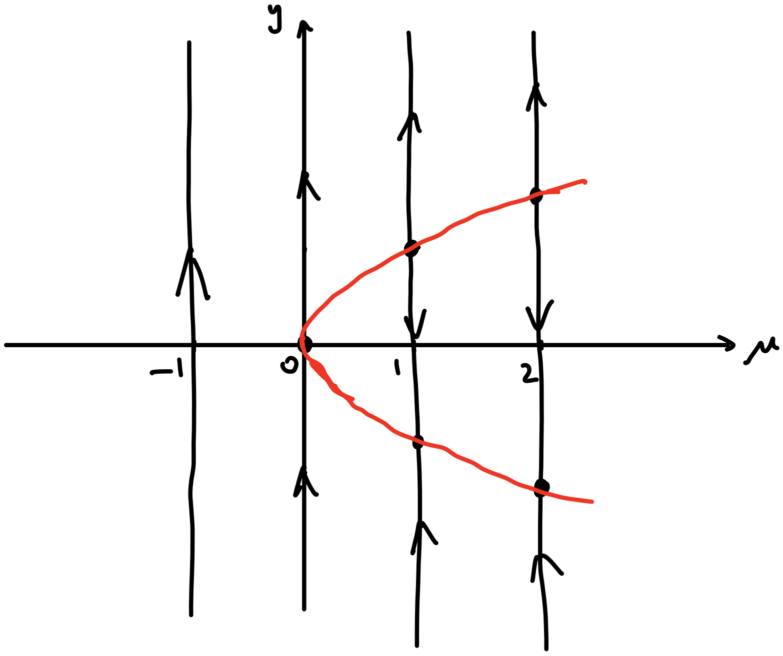
the discriminant "$b^2 - 4ac$" $= 0$

Applied to $y^2 - \mu = 1 \cdot y^2 + 0 \cdot y + (-\mu)$

$$b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-\mu) = 4\mu = 0 \quad \Rightarrow \quad \mu = 0.$$

- Bifurcation diagram: sketch phase lines in (μ, y)-plane

  connect all equilibrium pts for every μ as a curve.



Ex Sketch bifurcation diagram of

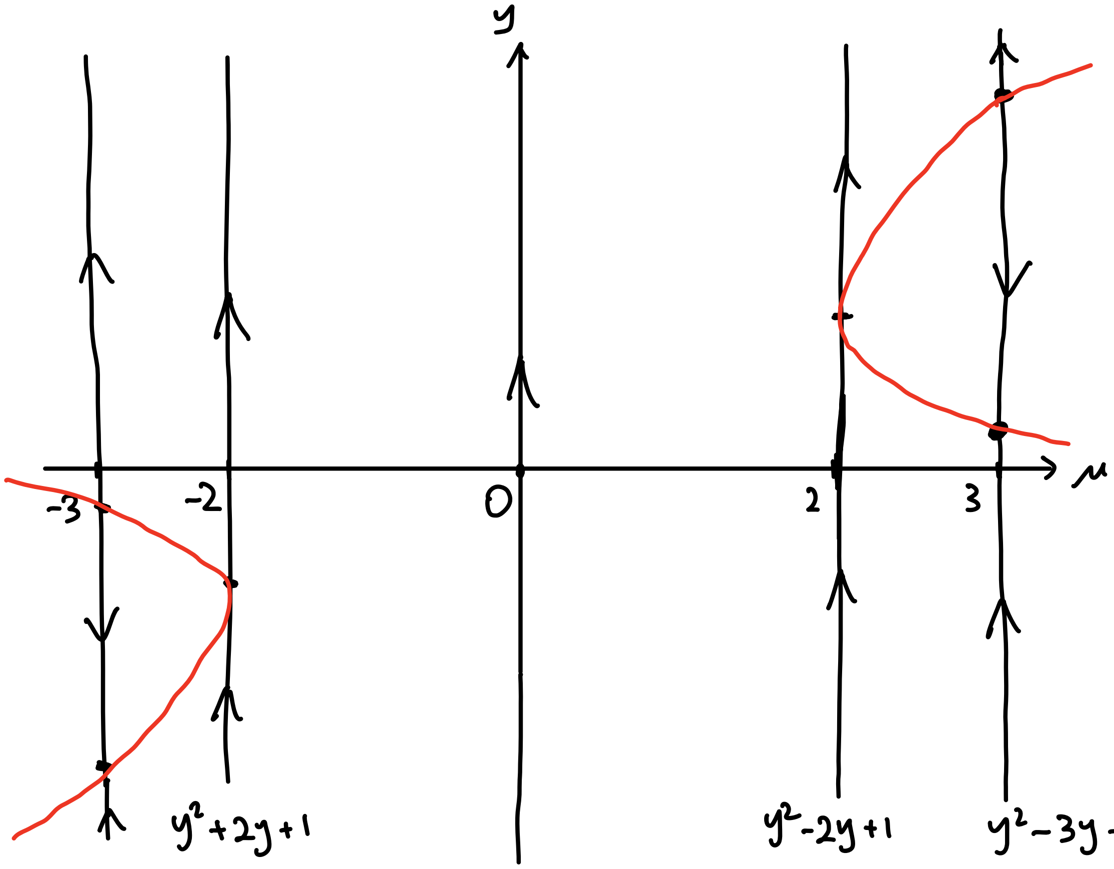
$$\frac{dy}{dt} = y^2 - \mu y + 1$$

To find bif. values,

$$(-\mu)^2 - 4 \cdot 1 \cdot 1 = 0$$

$$\mu^2 = 4$$

$$\mu = \pm 2$$



$y^2+2y+1 = (y+1)^2$ $\quad y^2+1 \quad$ $y^2-2y+1 = (y-1)^2$ $\quad y^2-3y+1$

$y^2 - 3y + 1 = 0$

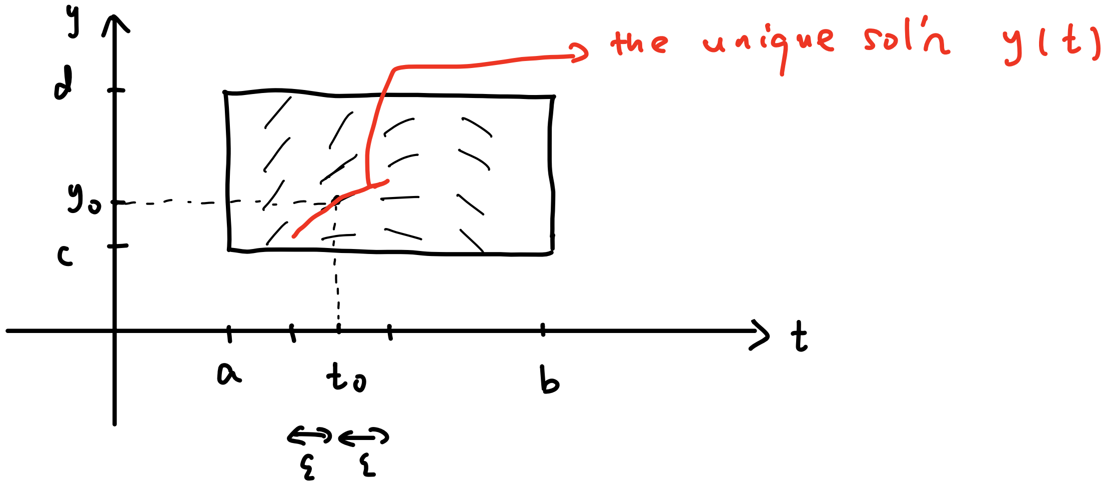
$y = \frac{3 \pm \sqrt{5}}{2}$

1.5 Existence and uniqueness of sol'ns

Consider an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad \text{------ ... (1)}$$

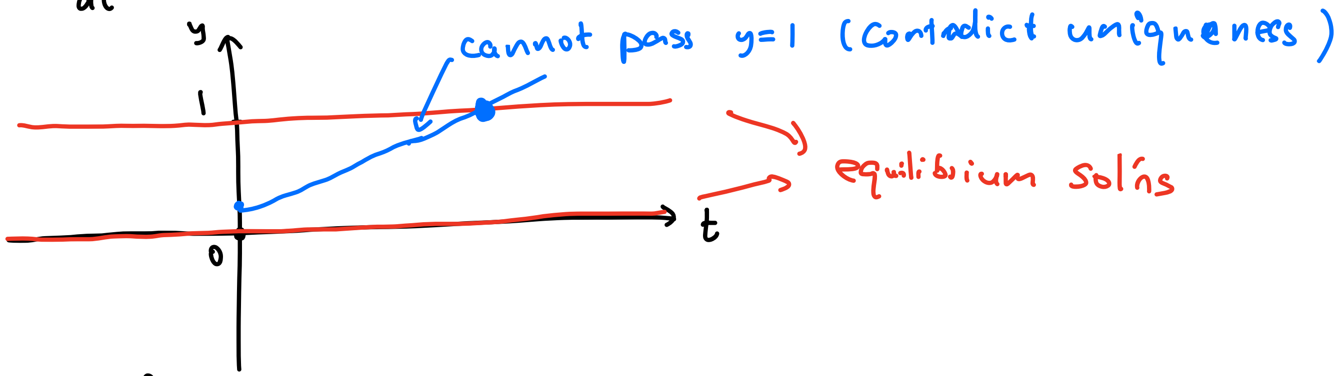
<u>Thm</u> Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in $\{(t,y) \mid a<t<b,\ c<y<d\}$. If (t_0, y_0) is a point in it, then there exists $\varepsilon > 0$ such that there exists a unique sol'n $y(t)$ to (1) defined on $(t_0 - \varepsilon, t_0 + \varepsilon)$.



- Existence : you can always find a sol'n
- Uniqueness : you can only find one sol'n .

  In other words , if you find two sol'ns , they must be the same.
- By applying thm iteratively , one can extend the sol'n to the boundary of the rectangle .
- Consequence of uniqueness : sol'ns cannot cross each other

<u>Ex</u> $\frac{dy}{dt} = y(1-y)$



- Why $\frac{\partial f}{\partial y}$?