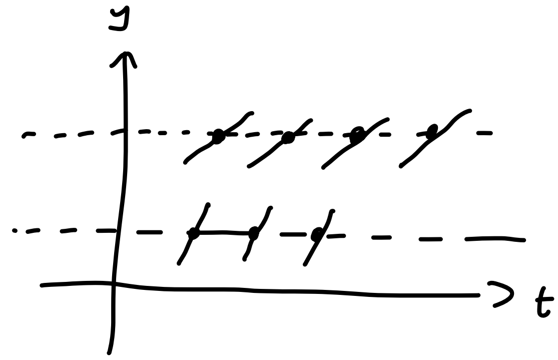
1.6 Equilibria and the phase line

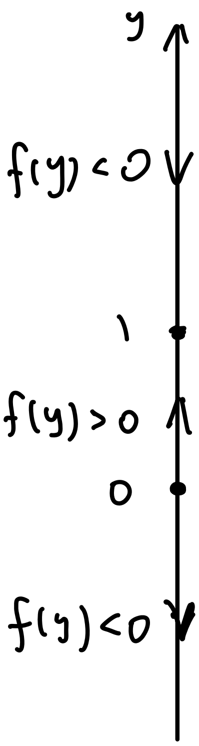
Consider an autonomous DE

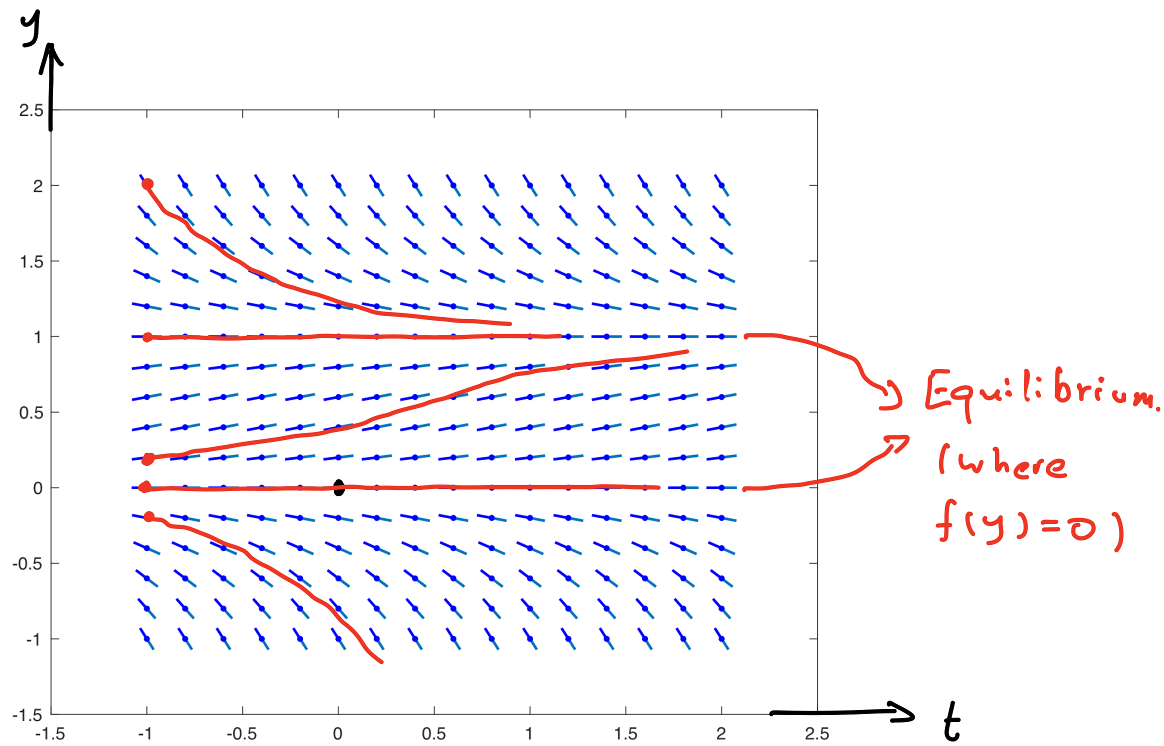
$$\frac{dy}{dt} = f(y)$$



In its slope field, along each horizontal line, the slopes are the same.

Ex $\quad \frac{dy}{dt} = y(1-y)$





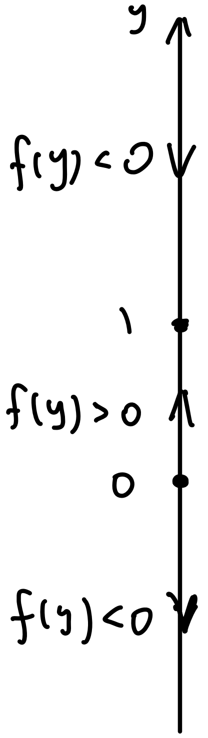
Phase line : a vertical y-axis, showing the behavior of the sol'ns.

① Identify all equilibrium pts (that is, where $f(y)=0$)

② Identify intervals where $f(y)>0$, draw "↑"

③ Identify intervals where $f(y) < 0$, draw "$\downarrow$"

Test by picking special pts in the intervals.



Info. we obtain :

- If initial value is 0 or 1, then sol'n is constant.
- If initial value is in $(-\infty, 0)$, then sol'n is decreasing, approaching $-\infty$.
- If initial value is in $(0, 1)$, then sol'n is increasing, approaching 1.
- If initial value is in $(1, \infty)$, then sol'n is decreasing, approaching 1.

Ex Sketch phase line for

$$\frac{dy}{dt} = \frac{e^{-y}(y^4 - 4y^2)}{y^2 + 2}$$

$(-3)^4 - 4 \times (-3)^2 = 81 - 36 > 0$

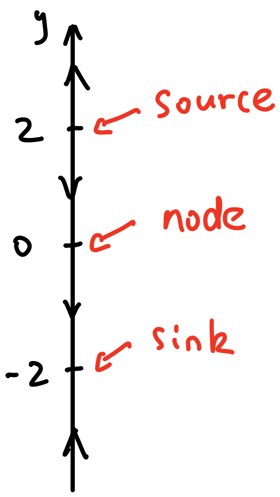
$(-1)^4 - 4 \times (-1)^2 = 1 - 4 < 0$

Determine behavior of sol'ns to initial value problems w/

① $y(0) = 1$

② $y(0) = -3$

(inc. / dec. ? , approaching ?)



Equilibrium pts :

$$\frac{e^{-y}(y^4 - 4y^2)}{y^2+2} = 0$$

$$e^{-y}(y^4 - 4y^2) = 0$$

$$y^4 - 4y^2 = 0$$

$$y^2(y^2-4) = 0$$

$$y^2 = 0 \quad \text{or} \quad y^2 = 4$$

$$y = 0,\ 2,\ -2$$

$f(-3) > 0$

$f(-1) < 0$

$f(1) < 0$

$f(3) > 0$

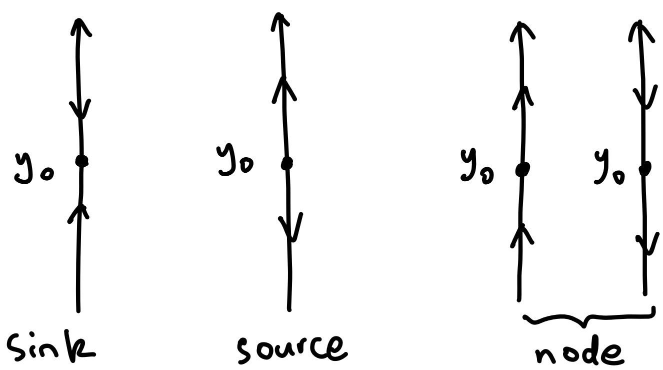
Initial value problem w/ $y(0) = 1$: Sol'n is dec., approaching 0.

Initial value problem w/ $y(0) = -3$: Sol'n is inc., approaching -2

Classification of equilibrium pts

Let y_0 be an equilibrium pt.

- We say y_0 is a sink if any sol'n w/ initial value sufficiently close to y_0 will approach y_0 as t increases.
- We say y_0 is a source if any sol'n w/ initial value sufficiently close to y_0 will get away from y_0 as t increases.
- Otherwise, we say y_0 is a node

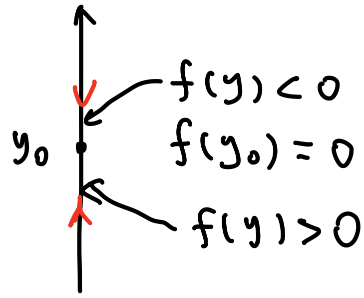


Thm Let y_0 be an equilibrium pt of $\frac{dy}{dt} = f(y)$

- If $f'(y_0) < 0$, then y_0 is a sink.

- If $f'(y_0) > 0$, then y_0 is a source.

Case $f'(y_0) < 0$:



- If $f'(y_0) = 0$, then no info is obtained.

Ex $\frac{dy}{dt} = \sin y + y^2 - \pi^2$

① Verify that $y = \pi$ is an equilibrium pt.

② Determine its type (sink/source/node)

① $\sin \pi + \pi^2 - \pi^2 = 0 \Rightarrow y = \pi$ is an equilibrium pt.

② $f'(y) = \cos y + 2y$

$f'(\pi) = \cos \pi + 2\pi = -1 + 2\pi > 0$

$\Rightarrow y = \pi$ is a source.

Ex $\frac{dy}{dt} = y^4 - 4y^2$ has equilibrium pts $-2, 0, 2$.

Apply f' test to determine types.

$f'(y) = 4y^3 - 8y$

$f'(-2) = 4 \cdot (-2)^3 - 8 \cdot (-2) = -32 + 16 = -16 < 0 \Rightarrow -2$ is a sink

$f'(0) = 4 \cdot 0^3 - 8 \cdot 0 = 0$ "inconclusive"

$f'(2) = 4 \cdot 2^3 - 8 \cdot 2 = 32 - 16 = 16 > 0 \Rightarrow 2$ is a source.