1.8 (continued)

A first order linear DE:

$$\frac{dy}{dt} = a(t)\,y + b(t) \qquad \text{------ --- (1)}$$

A first order linear DE is <u>homogeneous</u> if it has the form

$$\frac{dy}{dt} = a(t)\,y \qquad \text{--------- (2)}$$

<u>Thm</u> (linear principle) If $y_1(t)$, $y_2(t)$ are sol'ns to (2), and k is any constant, then

- $y_1(t) + y_2(t)$ is also a sol'n to (2)
- $k\,y_1(t)$ is also a sol'n to (2)

(That is, all sol'ns to (2) form a vector space).

<u>Proof of item 1</u>

$$\frac{dy_1}{dt} = a(t)\,y_1 \qquad \frac{dy_2}{dt} = a(t)\,y_2$$

$$\Rightarrow \quad \frac{d}{dt}(y_1 + y_2) = a(t)\,y_1 + a(t)\,y_2 = a(t)\left(y_1 + y_2\right)$$

$\Rightarrow$ $y_1 + y_2$ is a sol'n to (2).

<u>Thm</u> Suppose $y_p(t)$ is a particular sol'n to (1). Then any sol'n to (1) has the form

$$y(t) = y_p(t) + y_h(t) \qquad \text{------ (3)}$$

where $y_h(t)$ is any sol'n to (2).

In principle, to get general sol'n to (1), we can proceed as

$$\begin{cases} \text{find a particular sol'n } y_p(t) \text{ to } (1) \\ \text{find general sol'n } y_h(t) \text{ to } (2) \end{cases}$$

<u>Proof</u> • Prove that (3) is a sol'n to (1)

$$\frac{d}{dt}\left(y_p(t) + y_h(t) \right) = \frac{dy_p}{dt} + \frac{dy_h}{dt}$$

$$= a(t)\, y_p + b(t) + a(t)\, y_h$$

$$= a(t)(y_p + y_h) + b(t)$$

$\Rightarrow$ $y_p + y_h$ is a sol'n to (1)

• Prove that any sol'n to (1) has the form (3)

Suppose $y(t)$ is a sol'n to (1). Write

$$y(t) = y_p(t) + (y(t) - y_p(t))$$

$$\frac{d}{dt}(y(t) - y_p(t)) = \frac{dy}{dt} - \frac{dy_p}{dt}$$

$$= (a(t)\, y + b(t)) - (a(t)\, y_p + b(t))$$

$$= a(t) (y - y_p)$$

$\Rightarrow$ $y - y_p$ is a sol'n to (2)

Last time: $\frac{dy}{dt} + \frac{1}{t}y = t^2$

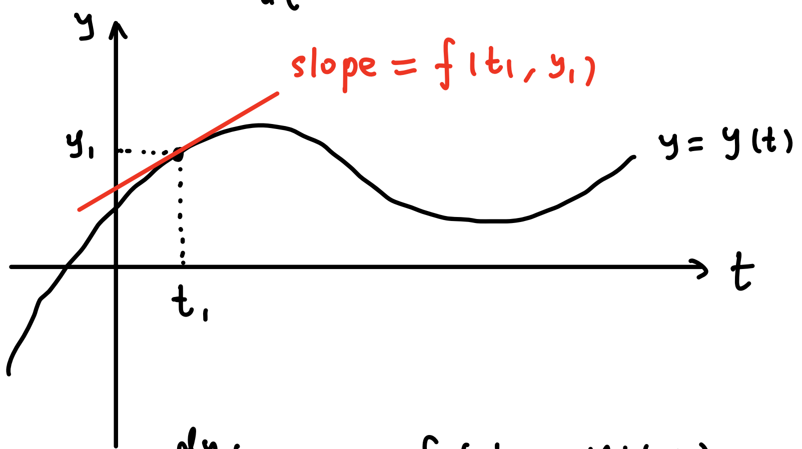
General sol'n $\quad y = \underbrace{\frac{1}{4}t^3}_{y_p(t)} + \underbrace{\frac{C}{t}}_{y_h(t)}$

homogeneous eq. $\frac{dy}{dt} + \frac{1}{t}y = 0$

$$\frac{d}{dt}\left(\frac{C}{t}\right) + \frac{1}{t}\left(\frac{C}{t}\right) = C\cdot\left(-\frac{1}{t^2}\right) + \frac{C}{t^2} = 0$$

1.3 Slope fields

Consider $\frac{dy}{dt} = f(t, y)$. Let $y(t)$ be a sol'n.



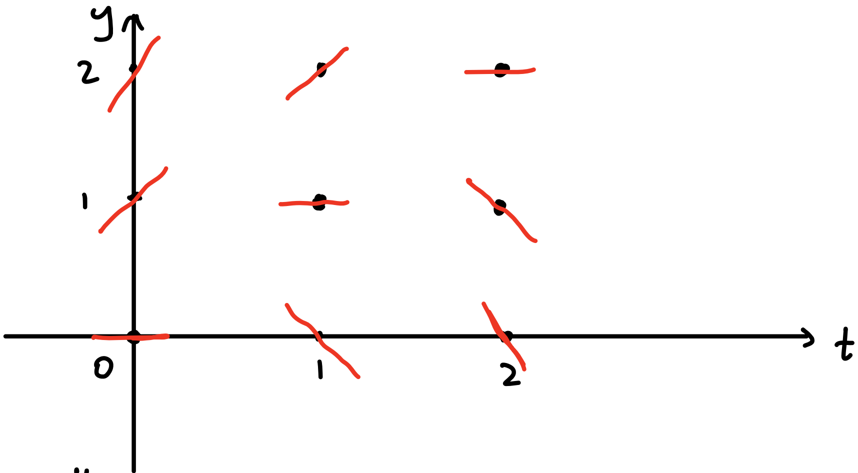
$y_1 = y(t_1)$

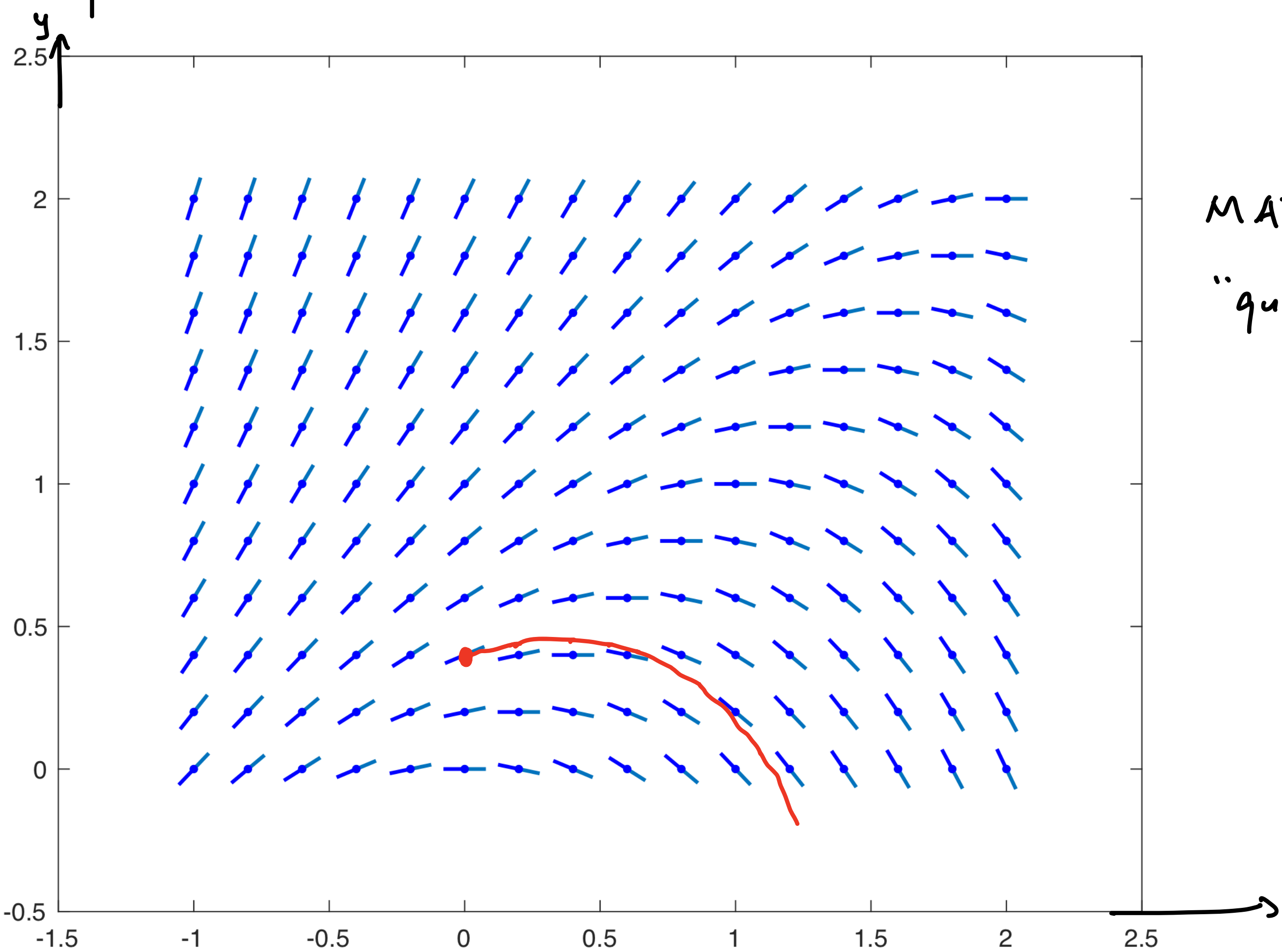
$$\frac{dy}{dt}(t_1) = f(t_1, y(t_1)) = f(t_1, y_1)$$

Slope field of $f(t, y)$: At each point (t, y), draw a small line segment w/ slope $f(t, y)$.

Any sol'n to $\frac{dy}{dt} = f(t, y)$ is tangent to these segments whenever it passes the corresponding point (t, y)

Ex $\frac{dy}{dt} = y - t$





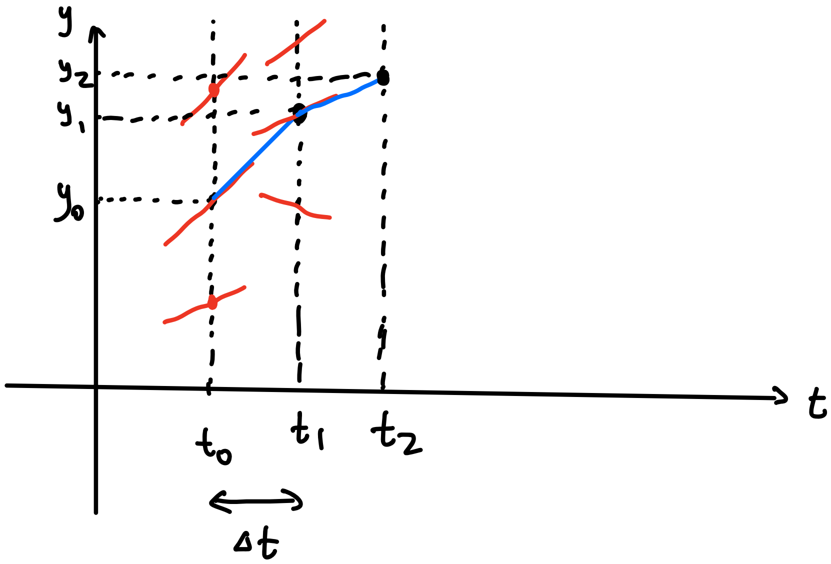
MATLAB
"quiver"

1.4 Euler's method

To solve the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

numerically, follow the slope field :



$$y_1 = y_0 + \Delta t \cdot f(t_0, y_0)$$

$$y_2 = y_1 + \Delta t \cdot f(t_1, y_1)$$

$$\vdots$$

$$t_k = t_0 + k\Delta t \qquad k = 0, 1, 2, \cdots$$

$$y_{k+1} = y_k + \Delta t \cdot f(t_k, y_k)$$

<u>Euler's method</u>

y_k is an approximation of $y(t_k)$

Ex Use Euler's method w/ $\Delta t = 0.5$ to solve

$$\frac{dy}{dt} = \underbrace{y - t}_{f(t,y)}, \quad y(0) = 0.4$$

to approximate $y(1)$

$$\#\text{ of time steps} = \frac{1-0}{0.5} = 2$$

$$t_0 = 0, \quad t_1 = 0.5, \quad t_2 = 1$$

$$y_1 = y_0 + \Delta t \cdot f(t_0, y_0) = 0.4 + 0.5 \cdot f(0, 0.4)$$

$$= 0.4 + 0.5 \cdot (0.4 - 0) = 0.6$$

$$y_2 = y_1 + \Delta t \cdot f(t_1, y_1) = 0.6 + 0.5 \cdot f(0.5, 0.6)$$

$$= 0.6 + 0.5 \cdot (0.6 - 0.5) = \boxed{0.65}$$

Error of Euler's method

- For a fixed time interval, smaller Δt gives better approximation (although requires more calculation).

  The error of the Euler's method is about proportional to Δt.

- For a fixed Δt, if one runs the method for a long time, the error usually grows exponentially in time.