1.8, 1.9 Linear equations

Def A first order DE for $y(t)$ is linear if it has the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

- Solve linear DE by integrating factors.

Denote $g(t) = -a(t)$. Then

$$\frac{dy}{dt} + g(t)y = b(t)$$

Multiply by the integrating factor

$$\boxed{\mu(t) = e^{\int g(t)\,dt}}$$

$\mu(t)$ satisfies

$$\mu'(t) = e^{\int g(t)\,dt} \cdot g(t)$$

$$= \mu(t) \cdot g(t)$$

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

$$\frac{d}{dt}(\mu(t)y)$$

$$= \mu'(t)y + \mu(t)\frac{dy}{dt}$$

$$= \mu(t)g(t)y + \mu(t)\frac{dy}{dt}$$

$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t)$$

Integrate on both sides

$$\mu(t)y = \int \mu(t)b(t)\,dt$$

$$y = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt$$

- When calculating $\int g(t)\,dt$ in the integrating factor,
  - You don't need "$+C$"
  - If it involves $\ln|\cdots|$, usually you don't need the absolute value.

<u>Ex</u> Solve $\dfrac{dy}{dt} + \underbrace{\dfrac{1}{t}}_{}\, y = t^2$

$\hookrightarrow g(t) = \dfrac{1}{t}$

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t$$

$$t\frac{dy}{dt} + y = t^3$$

$$\frac{d}{dt}(t\,y) = t^3$$

$$t\,y = \int t^3\,dt = \frac{1}{4}t^4 + C$$

$$y = \frac{1}{4}t^3 + \frac{C}{t}$$

<u>Ex</u> Solve initial value problem

$$\frac{dy}{dt} = \frac{1}{t}y + t, \quad y(1) = 2$$

$$\frac{dy}{dt} \underbrace{-\ \frac{1}{t}}_{}\, y = t$$

$\hookrightarrow g(t) = -\dfrac{1}{t}$

$$\mu(t) = e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1} = \frac{1}{t}$$

$$\frac{1}{t}\cdot\frac{dy}{dt} - \frac{1}{t^2}y = 1$$

$$\frac{d}{dt}\left(\frac{1}{t} y\right) = 1$$

$$\frac{1}{t} y = \int 1 \, dt = t + C$$

$$y = t^2 + Ct$$

initial condition $y(1) = 2$

$$2 = 1^2 + C \cdot 1 \qquad C = 1$$

$\Rightarrow$ Sol'n $\quad y = t^2 + t$

<u>Ex</u> Solve $\frac{dy}{dt} = ty + t^3$

$$\frac{dy}{dt} - ty = t^3$$

$$g(t) = -t$$

$$\mu(t) = e^{\int -t\,dt} = e^{-\frac{1}{2}t^2}$$

$$e^{-\frac{1}{2}t^2} \frac{dy}{dt} - e^{-\frac{1}{2}t^2} t y = e^{-\frac{1}{2}t^2} t^3$$

$$\frac{d}{dt}\left(e^{-\frac{1}{2}t^2} y\right) = e^{-\frac{1}{2}t^2} t^3$$

$$e^{-\frac{1}{2}t^2} y = \int e^{-\frac{1}{2}t^2} t^3 \, dt$$

$$\int e^{-\frac{1}{2}t^2} t^3 \, dt \qquad w = -\frac{1}{2}t^2$$

$$= \int e^w \cdot 2w \, dw \qquad dw = -t\,dt$$

$$= 2 \int e^w w \, dw \qquad u = w \qquad v = e^w$$

$$= 2\left(w e^w - \int e^w dw\right) \qquad du = dw \qquad dv = e^w dw$$

$$= 2(we^w - e^w) + C$$
$$= 2(-\tfrac{1}{2}t^2 e^{-\frac{1}{2}t^2} - e^{-\frac{1}{2}t^2}) + C$$

$$e^{-\frac{1}{2}t^2} y = 2(-\tfrac{1}{2}t^2 e^{-\frac{1}{2}t^2} - e^{-\frac{1}{2}t^2}) + C$$
$$= -t^2 e^{-\frac{1}{2}t^2} - 2e^{-\frac{1}{2}t^2} + C$$
$$y = -t^2 - 2 + Ce^{\frac{1}{2}t^2}$$

Ex A vat initially contains 10 gallons salty water w/ concentration 3 oz/gallon. Salty water w/ concentration 1 oz/gallon flows in at a rate of 4 gallon/sec. Well-mixed liquid flows out at a rate of 6 gallon/sec. What is the concentration of salt in the vat when it contains 4 gallons liquid?

Volume: $V(t) = 10 - 2t$

Amount of salt: $S(t)$ concentration: $\dfrac{S(t)}{10-2t}$

$$\frac{dS}{dt} = \underbrace{1 \times 4}_{\text{in flow}} - \underbrace{6 \cdot \frac{S(t)}{10-2t}}_{\text{out flow}} = 4 - \frac{3}{5-t} S, \quad S(0) = 30$$

$$\frac{dS}{dt} + \frac{3}{5-t} S = 4$$

$$g(t) = \frac{3}{5-t}$$

$$\mu(t) = e^{\int \frac{3}{5-t} dt} = e^{-3\ln(5-t)} = e^{\ln((5-t)^{-3})}$$
$$= (5-t)^{-3} = \frac{1}{(5-t)^3}$$

$$\frac{1}{(5-t)^3}\frac{dS}{dt} + \frac{3}{(5-t)^4}S = 4\cdot\frac{1}{(5-t)^3}$$

$$\frac{d}{dt}\left(\frac{1}{(5-t)^3}S\right) = 4\cdot\frac{1}{(5-t)^3}$$

$$\frac{1}{(5-t)^3}S = 4\int\frac{1}{(5-t)^3}\,dt$$

$$= \frac{2}{(5-t)^2} + C$$

$$\int\frac{1}{(5-t)^3}dt \qquad u=5-t$$

$$= -\int\frac{1}{u^3}du \qquad du=-dt$$

$$= -\frac{1}{u^2}\cdot\frac{1}{-2} + C$$

$$= \frac{1}{2}\cdot\frac{1}{(5-t)^2} + C$$

$$S = 2(5-t) + C(5-t)^3$$

Initial condition: $S(0) = 30$

$$\frac{1}{(5-0)^3}\cdot 30 = \frac{2}{(5-0)^2} + C$$

$$C = \frac{30}{125} - \frac{2}{25} = \frac{4}{25}$$

$$\Rightarrow\quad S = 2(5-t) + \frac{4}{25}(5-t)^3$$

When $V = 10-2t = 4$, $t=3$

$$S(3) = 2(5-3) + \frac{4}{25}(5-3)^3 = 4 + \frac{4}{25}\cdot 8 = \frac{132}{25}$$

Concentration: $\dfrac{\frac{132}{25}}{4} = \dfrac{33}{25}$.

- Higher order linear DE: (p-th order)

$$\frac{d^p y}{dt} + a_{p-1}(t)\frac{d^{p-1}y}{dt^{p-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)\,y = b(t)$$

- System of <u>linear</u> DEs : (first order)

$$\begin{cases} \dfrac{dy_1}{dt} = a_{11}(t)\, y_1(t) + a_{12}(t)\, y_2(t) + \cdots + a_{1n}(t)\, y_n(t) + b_1(t) \\ \cdots \quad - \quad - \quad - \;. \\ \dfrac{dy_n}{dt} = a_{n1}(t)\, y_1(t) + \quad \cdots \quad + a_{nn}(t)\, y_n(t) + b_n(t) \end{cases}$$