1.2 Separation of variables

A first order DE is <u>separable</u> if it has the form

$$\frac{dy}{dt} = g(t)\,h(y)$$

To solve it,

$$dy = g(t)\,h(y)\,dt$$

$$\frac{dy}{h(y)} = g(t)\,dt$$

$$\int \frac{dy}{h(y)} = \int g(t)\,dt$$

Ex Find general sol'n of $\frac{dy}{dt} = \underbrace{\frac{t^2}{y^3}}_{t^2 \cdot \frac{1}{y^3}}$

$$y^3\,dy = t^2\,dt$$

$$\int y^3\,dy = \int t^2\,dt$$

$$\cancel{\frac{1}{4}y^4 + C_1 = \frac{1}{3}t^3 + C_2}$$

$$\frac{1}{4}y^4 = \frac{1}{3}t^3 + C$$ remember to keep the C !

$$y^4 = 4\left(\frac{1}{3}t^3 + C\right)$$

$$y = \left(4\left(\frac{1}{3}t^3 + C\right)\right)^{\frac{1}{4}}$$

Ex Solve initial value problem

$$\frac{dy}{dt} = (y^2+1)\,t, \qquad y(0) = 1$$

$$\frac{dy}{y^2+1} = t\,dt$$

$$\int \frac{dy}{y^2+1} = \int t\,dt$$

$$\tan^{-1} y = \frac{1}{2}t^2 + C$$

$$y = \tan\left(\frac{1}{2}t^2 + C\right)$$ general sol'n

Substitute $t=0$, $y=1$,

$$1 = \tan\left(\cancel{\tfrac{1}{2}\cdot 0^2} + C\right) \qquad \text{Take } C = \frac{\pi}{4}$$

$$\Rightarrow \quad y = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)$$

- In a separable DE, $\frac{dy}{dt} = g(t)\,h(y)$, if $h(y) = 0$ has a sol'n $y = y_*$, then the constant function $y(t) = y_*$ is also a sol'n to the DE.

Ex $\dfrac{dy}{dt} = (y-1)^2 \cos t$ general sol'n.

Missing sol'n: $(y-1)^2 = 0$ $\quad \underline{y = 1}$

$$\frac{dy}{(y-1)^2} = \cos t \, dt$$

$$\int \frac{dy}{(y-1)^2} = \int \cos t \, dt$$

$$-\frac{1}{y-1} = \sin t + C$$

$$\frac{1}{y-1} = -(\sin t + C)$$

$$y - 1 = -\frac{1}{\sin t + C}$$

$$y = 1 - \frac{1}{\sin t + C}$$

$$\int \frac{dy}{(y-1)^2} \qquad u = y-1$$
$$= \int \frac{du}{u^2} \qquad du = dy$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{y-1} + C$$

General sol'n: $y = 1 - \dfrac{1}{\sin t + C}$ or $y = 1$

- Sometimes after taking integral, y cannot be solved from the algebraic eq. One can only express the sol'n as an implicit function.

Ex $\dfrac{dy}{dt} = \dfrac{t}{y^4 + 1}$ general sol'n.

$$(y^4 + 1)\, dy = t \, dt$$

$$\frac{1}{5} y^5 + y = \frac{1}{2} t^2 + C$$

y cannot be solved explicitly. The general sol'n is expressed as an implicit function.

Ex Solve exponential growth model $\frac{dP}{dt} = kP$

Missing sol'n: $P = 0$

$$\frac{dP}{P} = k\,dt$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$$|P| = e^{C_1 + kt} = e^{C_1} e^{kt}$$

$$P = \underbrace{\pm e^{C_1}}_{\text{a constant } C \text{ which is } \neq 0.} e^{kt}$$

General sol'n: $P = C e^{kt}$ where C is any constant.

Ex Solve logistic model $\frac{dP}{dt} = P(1-P)$ (special case $k=1, N=1$)

for $0 < P < 1$.

$$\frac{1}{P(1-P)}\,dP = dt$$

$$\int \frac{1}{P(1-P)}\,dP = \int dt$$

$$\ln|P| - \ln|1-P| = t + C_1$$

$$\ln\left|\frac{P}{1-P}\right| = t + C_1$$

$$\left|\frac{P}{1-P}\right| = e^{C_1} e^{t}$$

Since $0 < P < 1$,

$$\frac{P}{1-P} = e^{C_1} e^{t} = C e^{t}$$

$C = e^{C_1} > 0$.

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$1 = A(1-P) + BP$$

$P = 0 \rightsquigarrow 1 = A\cdot(1-0) = A$

$P = 1 \rightsquigarrow 1 = A\cdot(1-1) + B\cdot 1 = B$

$$\int \frac{1}{P(1-P)}\,dP = \int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP$$

$$= \ln|P| - \ln|1-P| + C$$

$$P = Ce^t(1-P) = Ce^t - Ce^tP$$

$$(1+Ce^t)P = Ce^t$$

$$P = \frac{Ce^t}{1+Ce^t} = \frac{1}{\frac{1}{Ce^t}+1} \qquad C > 0$$

One application : as $t \to \infty$, $P(t) \longrightarrow 1$

- A DE of the form $\frac{dy}{dt} = h(y)$ is called an <u>autonomous</u> DE.

  It's a special case of separable DE.

- Mixing problems

<u>Ex</u> A vat initially contains 10 gallons pure water.
Salty water w/ concentration 3oz/gallon starts to flow in at a rate of 2gallon/sec.
Well-mixed liquid flows out to make the liquid volume in the vat unchanged.
What is the concentration of salt in the vat after 4 sec ?

Denote the amount of salt in the vat as $S(t)$

Concentration in the vat : $\frac{S(t)}{10}$

$$\frac{dS}{dt} = \underbrace{3 \times 2}_{\text{in flow}} - \underbrace{\frac{S(t)}{10} \times 2}_{\text{out flow}} = 6 - \frac{1}{5}S, \qquad S(0) = 0$$

$$\frac{dS}{6-\frac{1}{5}S} = dt$$

$$\int \frac{dS}{6-\frac{1}{5}S} = \int dt$$

$$-5\ln\left|6-\frac{1}{5}S\right| = t + C$$

Substitute $t=0,\ S=0$

$$-5\ln\left|6 \cancel{-\tfrac{1}{5}\cdot 0}\right| = \cancel{0+}\ C$$

$$C = -5\ln 6$$

$$\leadsto -5\ln\left(6-\frac{1}{5}S\right) = t - 5\ln 6$$

$$\ln\left(6-\frac{1}{5}S\right) = -\frac{1}{5}t + \ln 6$$

$$6-\frac{1}{5}S = 6e^{-\frac{1}{5}t}$$

$$-\frac{1}{5}S = 6e^{-\frac{1}{5}t} - 6$$

$$S = -30e^{-\frac{1}{5}t} + 30$$

At $t=4$, $S(4) = -30e^{-\frac{4}{5}} + 30$

$\Rightarrow$ concentration is $\dfrac{S(4)}{10} = -3e^{-\frac{4}{5}} + 3$.

$$\int \frac{dS}{6-\frac{1}{5}S} \qquad u = 6-\frac{1}{5}S, \quad du = -\frac{1}{5}\,dS$$

$$= -5\int \frac{du}{u}$$

$$= -5\ln|u| + C$$

$$= -5\ln\left|6-\frac{1}{5}S\right| + C$$