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1.1 Modeling via differential equations

Consider a quantity which changes w/ time. The rate of change (derivatives) may depend on its current value.

Ex 1 Rat population in Athens $P(t)$.

Its rate of change (growth) is proportional to the current population. (say, coefficient is $k > 0$)

$$\rightsquigarrow \quad \frac{dP}{dt} = kP$$

General solution: $P(t) = Ce^{kt}$ (for any constant C)

$$\frac{d}{dt}(\underbrace{Ce^{kt}}_{P}) = Ce^{kt} \cdot k = kP$$

The function $P(t)$ can be determined if we know its initial value $P(t_0) = P_0$

$$P_0 = Ce^{kt_0} \qquad C = P_0 e^{-kt_0}$$

$\rightsquigarrow$ the unique sol'n satisfying the initial condition is

$$P(t) = P_0 e^{-kt_0} e^{kt} = P_0 e^{k(t-t_0)}$$ "particular solution"

If $P_0 > 0$, then the population grows exponentially in time.

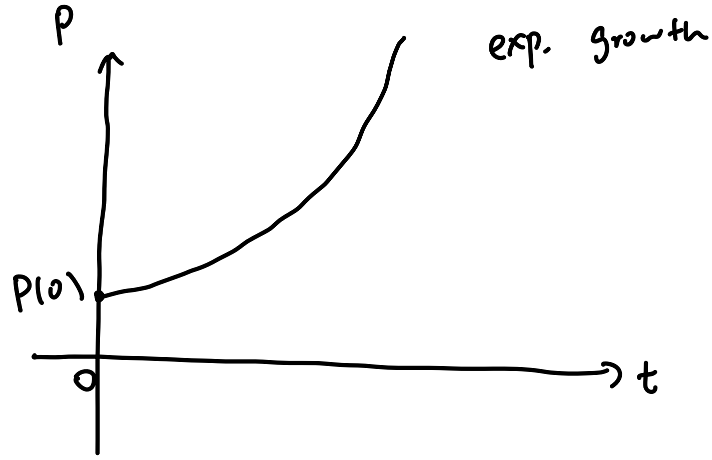
Ex 2 An improved model : logistic population model.

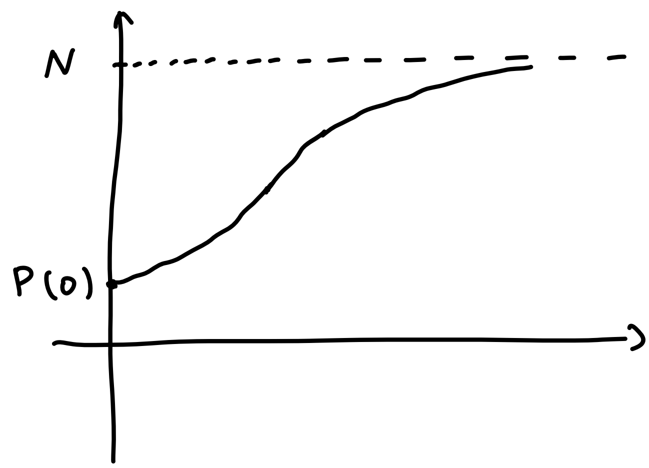
Population growth is restricted by the environment.

Constant parameter N : carrying capacity of the environment.

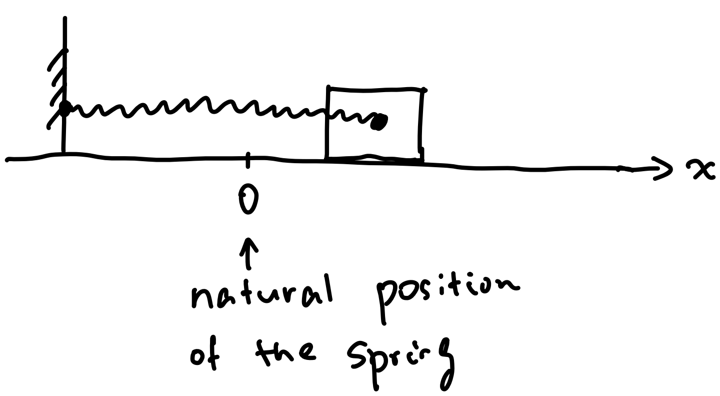
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

- when P is much less than N, then $\frac{dP}{dt} \approx kP$ (similar to exponential growth)
- when $P > N$, then $\frac{dP}{dt} < 0$ ⇝ population decays.





Ex 3 Harmonic oscillator



$x(t)$: position of the block

m : mass - - - - - - - - -

k : spring constant

Hooke's law: magnitude of force on block is kx

Newton's 2nd law:

$$m \frac{d^2 x}{dt^2} = -kx$$

"a second order differential eq."

$\uparrow$ acceleration

Ex 4 A chemical reaction $A + 2B \longrightarrow 4D$

$c_A(t)$, $c_B(t)$, $c_D(t)$: concentration of chemicals.

The reaction rate is proportional to $c_A c_B^2$ (say, coeff. is k)

$$\begin{cases} \dfrac{dc_A}{dt} = -k c_A c_B^2 \\[2ex] \dfrac{dc_B}{dt} = -2k c_A c_B^2 \\[2ex] \dfrac{dc_D}{dt} = 4k c_A c_B^2 \end{cases}$$

"a system of differential eq.s"

- Generally, an (ordinary) differential equation is an equation containing some unknown functions and their (possibly high order) derivatives.
- It is p-th order if the highest order derivative involved has order p.
- A (particular) solution to a DE is a formula of unknown functions so that the DE is satisfied.
- The general solution to a DE is a formula of unknown functions (involving parameters), which describe all sol'ns to the DE

General form of a first order DE w/ one unknown function $y(t)$

$$\frac{dy}{dt} = f(t, y) \quad \text{------} \quad (*)$$

Initial value problem : $(*)$ w/ initial condition $y(t_0) = y_0$

"Usually" such an initial value problem has a unique sol'n.

Ex Verify that $x(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$ is a sol'n to $m\frac{d^2x}{dt^2} = -kx$.

$$\frac{dx}{dt} = -\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$\frac{d^2x}{dt^2} = -\sqrt{\frac{k}{m}} \cdot \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\, t\right) = -\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

$m\frac{d^2x}{dt^2} = -k\cos\left(\sqrt{\frac{k}{m}}\, t\right)$, $-kx = -k\cos\left(\sqrt{\frac{k}{m}}\, t\right)$ are equal.

How to study DEs ?

- Find explicit sol'ns. (only possible for some special cases).
- Numerical sol'ns : get approximations of sol'ns via algorithms.
- Qualitative analysis

  (is the sol'n increasing / decreasing ?

  concave up / down ?

  behavior of sol'n as $t \to \infty$?)