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Problem 1: Consider the following numerical differentiation method:
(1) The first order method $(f(x+h)-f(x))/h$.
(2) The second order centered difference $(f(x+h)-f(x-h))/2h$.
(3) The fourth order centered difference. You need to figure out its formula by using polynomial interpolation at $x-2h, x-h, x, x+h, x+2h$ to approximate $f'(x)$.
Test the accuracy order of these methods by applying to $f(x)=\sin x$ at $x=1$ with various values of h: take $h=2^{-n}, n=1,2,\dots,40$. Use a log-log plot for error vs. h to observe the accuracy order by the slope. Explain what happens when h is extremely small.

Problem 2: Apply composite trapezoid and composite Simpson rules to approximate the integral $\int_0^2 e^{-x}dx$. Check the accuracy order as in Problem 1 with $h=2^{-n}, n=1,2,\dots,10$.

Problem 3: Write a code to find the Gaussian quadrature nodes and weights as accurate as possible, on $[0,1]$ with $n=10$ and $w(x)=1$. Use it to approximate the integral $\int_0^1 \frac{4}{1+x^2}dx$ to give an approximation of π.

Problem 4: Consider the ODE

$$x' = \sin(t)\cos(x), \quad x(0)=0.2$$

Solve it numerically to approximate $x(1)$ by: forward Euler method, second order Taylor method, any second order Runge-Kutta method and any fourth order Runge-Kutta method. Check the accuracy order by using various h. (Here the exact solution is not available: to get the error of a numerical solution, you can use the same method with $h/2$ as a reference solution.)

Problem 5: The harmonic oscillator $x''=-x$ can be written into a system of two first-order ODEs

$$\begin{cases} x' = v \\ v' = -U'(x) \end{cases}, \quad \text{with } U(x)=\frac{1}{2}x^2$$

The energy $E=\frac{1}{2}v^2+U(x)$ is conserved along the solution. The solution oscillates periodically in time with a fixed magnitude.
(1) Take the initial condition $x(0)=0.8, v(0)=0.6$. Apply the forward Euler method with reasonably small h, and observe how the numerical solution deviates from the exact solution when t gets large (you can show that by plotting numerical/exact solution of $x(t)$ in a picture).
(2) The *symplectic Euler method* is

$$\begin{cases} x_{i+1} = x_i + hv_{i+1} \\ v_{i+1} = v_i - hU'(x_i) \end{cases}$$

Repeat (1) with the symplectic Euler method and explain why it behaves better than the forward Euler method in terms of the long time behavior.
(One theoretical explanation is that the energy $E_n^h := \frac{1}{2}v_n^2 + \frac{1}{2}x_n^2 - \frac{1}{2}v_n x_n$ is conserved along the numerical solution.)
(3) Repeat (1)(2) for the pendulum equation $x''=-\sin(x)$.