Stability of multistep methods (for stiff problems)

$$a_k x_i + \cdots + a_0 x_{i-k} = h\left(b_k f_i + \cdots + b_0 f_{i-k}\right) \qquad f_j := f(t_j, x_j)$$

Apply to $x' = \lambda x$,

$$a_k x_i + \cdots + a_0 x_{i-k} = h\lambda\left(b_k x_i + \cdots + b_0 x_{i-k}\right)$$

$$(a_k - h\lambda b_k) x_i + \cdots + (a_0 - h\lambda b_0) x_{i-k} = 0$$

Try $x_i = w^i \qquad w \in \mathbb{C}$

$$\underbrace{(a_k - h\lambda b_k) w^k + \cdots + (a_0 - h\lambda b_0) w^0}_{\phi(h\lambda,\, w)} = 0$$

The stability region is $\{ z \in \mathbb{C} : \phi(z, \cdot) \text{ has all zeros w/ } |\cdot| \le 1 \}$.

It is A-stable if the stability region contains $\{ z \in \mathbb{C} : \mathrm{Re}(z) < 0 \}$.

Ex 2nd order BDF $\quad (a_0, a_1, a_2) = (\frac{1}{2}, -2, \frac{3}{2}) \quad$ A-stable

$\qquad (b_0, b_1, b_2) = (0, 0, 1)$

Multistep methods w/ accuracy order > 2 cannot be A-stable

"Dahlquist barrier"

Boundary value problems

$u = u(x)$ on $[0, 1]$

$$u'' = f(x, u, u'), \quad u(0) = \alpha, \quad u(1) = \beta.$$

- One should think of $u'' = f(x, u, u')$ as a 1D steady state PDE where x is a spatial variable.

- Existence / uniqueness of sol'ns are generally not known, even for nice f. Only known for some special cases.

For example, given nice $p(x) > 0$, $q(x)$, $g(x)$, consider

$$\overbrace{\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u}^{Lu} = g(x) \qquad Lu = g$$

It has a unique sol'n by Sturm-Liouville theory.

Finite difference method

uniform mesh : $x_i = i\,h \qquad h = \frac{1}{n}$.

u_i as an approximation of $u(x_i)$

Already know: $u_0 = \alpha$, $u_n = \beta$

At each grid pt x_i, $1 \le i \le n-1$, approximate derivatives in ODE by numerical differentiation. (say, centered difference)

$$u'' = f(x, u, u')$$

$$\Downarrow$$

$$\frac{1}{h^2}\left(u_{i+1} - 2u_i + u_{i-1}\right) = f\left(x_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right) \qquad 1 \le i \le n-1$$

$n-1$ equations in $n-1$ variables $u_1, \cdots, u_{n-1}$

$\rightsquigarrow$ expect to have sol'n in nice situations

put into nonlinear solver.

A simple example : $u'' = g(x)$

$$\left(u_{i+1} - 2u_i + u_{i-1}\right) = h^2 g(x_i) \qquad 1 \le i \le n-1$$

$$\begin{cases} u_2 - 2u_1 + \cancel{u_0}_{\alpha} = h^2 g(x_1) \\ u_3 - 2u_2 + u_1 = h^2 g(x_2) \\ \quad \cdots \\ u_{n-1} - 2u_{n-2} + u_{n-3} = h^2 g(x_{n-2}) \\ \cancel{u_n}_{\beta} - 2u_{n-1} + u_{n-2} = h^2 g(x_{n-1}) \end{cases} \rightsquigarrow A\vec{u} = \vec{b}$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} h^2 g(x_1) - \alpha \\ h^2 g(x_2) \\ \vdots \\ h^2 g(x_{n-2}) \\ h^2 g(x_{n-1}) - \beta \end{pmatrix}$$

sym. neg. def.

Finite element method

Strong

$$u'' = f(x, u, u'), \quad u(0) = \alpha, \quad u(1) = \beta.$$

$$\int_0^1 u'' \phi \, dx = \int_0^1 f(x, u, u') \phi \, dx$$

$$\hookrightarrow \cancel{u'\phi \Big|_{x=0}^{1}} - \int_0^1 u' \phi' \, dx$$

by choosing $\phi(0) = \phi(1) = 0$

$\rightsquigarrow$ Weak formulation: seek for $u \in H^1([0,1])$, $u(0) = \alpha$, $u(1) = \beta$, s.t.

Weak

$$-\int_0^1 u' \phi' \, dx = \int_0^1 f(x, u, u') \phi \, dx, \quad \forall \phi \in H_0^1([0,1])$$

$\hookrightarrow \phi(0) = \phi(1) = 0$

$$H^1([0,1]) = \{ \phi : \phi \in L^2, \phi' \in L^2 \}$$

Idea: use a finite dimensional subspace to approximate $H^1([0,1])$.

(for example: piecewise polynomial spaces, global polynomial spaces, trig. func. spaces, splines, $\cdots$)

Say, $V \subset \{ u \in H^1([0,1]) : u(0) = \alpha, u(1) = \beta \}$. $\Big\}$ same dim $= n$

$W \subset H_0^1([0,1])$

Then we seek for numerical sol'n $u_h \in V$ s.t.

$$\int_0^1 u_h' \phi_h' \, dx = \int_0^1 f(x, u_h, u_h') \phi_h \, dx \qquad \forall \phi_h \in W$$

By writing $u_h = v_0 + \sum_{i=1}^{n} c_i v_i$ where $\{v_1, \cdots, v_n\}$ is a basis of (shifted) V, and test against a basis $\{w_1, \cdots, w_n\}$ of W, we get

$$\int_0^1 (v_0 + \Sigma c_i v_i)' w_j \, dx = \int_0^1 f(x, v_0 + \Sigma c_i v_i, (v_0 + \Sigma c_i v_i)') w_j \, dx$$

$j = 1, \cdots, n$

which are n eqs. in n var. $c_1, \cdots, c_n$