Thm If $f(t,x)$ is Lipschitz in x and the exact sol'n $x(t) \in C^2$, then for fixed T, the numerical sol'n x_n by forward Euler w/ $h = \frac{T-t_0}{n}$ satisfies

$$|x(T) - x_n| \le C(T) \cdot h$$

Proof Denote $e_i = x(t_i) - x_i$

$$x_{i+1} = x_i + h f(t_i, x_i)$$

$$x(t_{i+1}) = x(t_i) + h f(t_i, x(t_i)) + \underbrace{O(h^2)}_{\text{depending on } \|x(t)\|_{C^2([t_0,T])}}$$

$$\Rightarrow \quad e_{i+1} = e_i + h\underbrace{\big(f(t_i, x(t_i)) - f(t_i, x_i)\big)}_{|\cdot| \le L \cdot |x(t_i) - x_i| = L\cdot|e_i|} + O(h^2)$$

L: Lip. const. of f

$$|e_{i+1}| \le |e_i| + hL\cdot|e_i| + \underbrace{C}_{\text{depending on } \|x(t)\|_{C^2([t_0,T])}} h^2$$

$e_0 =$

$$= (1+Lh)|e_i| + Ch^2$$

$$\begin{aligned}
|e_n| &\le (1+Lh)|e_{n-1}| + Ch^2 \\
&\le (1+Lh)\big((1+Lh)|e_{n-2}| + Ch^2\big) + Ch^2 \\
&\le \cdots \le Ch^2\big(1 + (1+Lh) + (1+Lh)^2 + \cdots + (1+Lh)^{n-1}\big) \\
&= Ch^2 \frac{(1+Lh)^n - 1}{(1+Lh) - 1} \\
&= h \cdot \frac{C}{L}\left((1+Lh)^{\frac{1}{Lh}\cdot Lhn} - 1\right) \\
&\le h \cdot \frac{C}{L}\left(e^{Lhn} - 1\right)
\end{aligned}$$

$(1+\alpha)^{1/\alpha} < e \quad \forall \alpha > 0$

$$\le h \cdot \underbrace{\frac{C}{L}\left(e^{L(T-t_0)} - 1\right)}_{C(T)}$$

exponential growth in T

"Gronwall inequality"

$$x' \le A x + \_\_\_$$

$$x(t) \le \_\_\_\_\_$$

8.2 Taylor series method

$$x'(t) = f(t, x(t))$$

$$x(t+h) = x(t) + h \underbrace{x'(t)}_{f(t,x(t))} + \frac{h^2}{2} x''(t) + \frac{h^3}{6} x'''(t) + \cdots$$

forward Euler L.T.E. $= O(h^2)$

2nd order Taylor L.T.E. $= O(h^3)$

$$x''(t) = \partial_t f(t, x(t)) + \partial_x f(t, x(t)) \cdot x'(t)$$

$$= \partial_t f(t, x(t)) + \partial_x f(t, x(t)) \cdot f(t, x(t))$$

2nd order Taylor method:

$$x_{i+1} = x_i + h f(t_i, x_i) + \frac{h^2}{2}\left[\partial_t f(t_i, x_i) + \partial_x f(t_i, x_i) \cdot f(t_i, x_i)\right]$$

$$x'''(t) = \partial_{tt} f(t, x(t)) + \partial_{tx} f(t, x(t)) \cdot x'(t)$$

$$+ f(t, x(t)) \cdot \left(\partial_{tx} f(t, x(t)) + \partial_{xx} f(t, x(t)) \cdot x'(t)\right)$$

$$+ \partial_x f(t, x(t)) \cdot \left(\partial_t f(t, x(t)) + \partial_x f(t, x(t)) \cdot x'(t)\right)$$

$\rightsquigarrow$ 3rd order Taylor method.

- Taylor methods are only applicable when partial derivatives of f are easily calculated (be careful about vector-valued ODEs !)

8.3 Runge-Kutta method

Forward Euler $\quad x^* = x_i + h f(t_i, x_i)$

Try: $\quad x_{i+1} = x_i + b_1 h f(t_i, x_i) + b_2 h f(t_i + h, x^*)$

Choose coeffs b_1, b_2 so that Taylor expansion of x_{i+1} and $x(t_{i+1})$ match up to $O(h^2)$ (L.T.E. $= O(h^3)$, 2nd order accuracy)

$$x(t_{i+1}) = x(t_i) + h x'(t_i) + \frac{h^2}{2} x''(t_i) + \cdots$$

$$= \boxed{x(t_i)} + \boxed{h f(t_i, x(t_i))} + \boxed{\frac{h^2}{2}\left(\partial_t f(t_i, x(t_i)) + \partial_x f(t_i, x(t_i)) \cdot f(t_i, x(t_i))\right)} + \cdots$$

$$x_{i+1} = x_i + b_1 h f(t_i, x_i) + b_2 h f(t_i + h, x_i + h f(t_i, x_i))$$

$$= x_i + \underline{b_1 h f(t_i, x_i)} + b_2 h \Big(\underline{f(t_i, x_i)} + \underline{\partial_t f(t_i, x_i) \cdot h}$$

$$+ \underline{\partial_x f(t_i, x_i) \cdot h f(t_i, x_i)} + O(h^2) \Big)$$

$$= \boxed{x_i} + \boxed{h f(t_i, x_i) \cdot (b_1 + b_2)}$$

$$+ \boxed{h^2 \cdot b_2 \cdot \left(\partial_t f(t_i, x_i) + \partial_x f(t_i, x_i) \cdot f(t_i, x_i)\right)} + O(h^3)$$

$$\begin{cases} b_1 + b_2 = 1 \\ b_2 = \frac{1}{2} \end{cases} \qquad b_1 = \frac{1}{2}$$

$\Rightarrow$ Heun's method (second order)

$$\begin{cases} x^* = x_i + h f(t_i, x_i) \\ x_{i+1} = x_i + \frac{1}{2} h f(t_i, x_i) + \frac{1}{2} h f(t_i + h, x^*) \end{cases}$$

General framework of Runge-Kutta

$$x^{(1)} = x_i$$

$$x^{(2)} = x_i + a_{21} h f(\underline{t_i}, x^{(1)}) \qquad c_2 := a_{21}$$

$$x^{(3)} = x_i + a_{31} h f(\underline{t_i}, x^{(1)}) + a_{32} h f(\underline{t_i + c_2 h}, x^{(2)}) \qquad c_3 := a_{31} + a_{32}$$

$$x^{(4)} = x_i + a_{41} h f(t_i, x^{(1)}) + a_{42} h f(t_i + c_2 h, x^{(2)})$$

$$+ a_{43} h f(t_i + c_3 h, x^{(3)})$$

$\cdots\cdots$

$$\begin{cases} x^{(j)} = x_i + h \sum_{k=1}^{j-1} \underline{a_{jk}} f(t_i + c_k h, x^{(k)}) & c_j = \sum_{k=1}^{j-1} a_{jk} \\ \qquad j = 1, \cdots, s & \\ x_{i+1} = x_i + h \sum_{k=1}^{s} \underline{b_k} f(t_i + c_k h, x^{(k)}) & \end{cases}$$

Butcher table :

$$\begin{array}{c|cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
c_2 & a_{21} & 0 & \cdots & 0 & 0 \\
c_3 & a_{31} & a_{32} & \ddots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & 0 \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & 0 \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

Heun's :

$$\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}$$