7.3 (continued)

Gaussian quadrature

$$\int_a^b f(x)\,w(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

exact for poly. w/ deg $\le 2n+1$

- $\sum_{i=0}^{n} A_i = \int_a^b w(x)\,dx$, $A_i > 0$

  $\hookrightarrow$ take $f \equiv 1$

  $\hookrightarrow$ Let $p(x) = \dfrac{q(x)}{x - x_i} \in \Pi_n$ $\qquad q(x) = \prod_{j=0}^{n} (x - x_j)$

  Then $p^2(x) \in \Pi_{2n}$ $\Rightarrow$ quadrature for p^2 is exact

$$0 < \int_a^b p^2(x)\,w(x)\,dx = \sum_{j=0}^{n} A_j\, p^2(x_j) \qquad p(x_j) = 0 \quad \forall j \ne i$$

$$= A_i \underbrace{p^2(x_i)}_{>0} \qquad \Rightarrow A_i > 0$$

- For any $f \in C[a,b]$, the Gaussian quadrature w/ $n+1$ nodes

$$\sum_{i=0}^{n} A_{n,i} f(x_{n,i}) \text{ converges to } \int_a^b f(x)\,w(x)\,dx \text{ as } n \to \infty.$$

Thm (error estimate) For $f \in C^{2n+2}[a,b]$, Gaussian quadrature satisfies

$$\int_a^b f(x)\,w(x)\,dx = \sum_{i=0}^{n} A_i f(x_i) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b q^2(x)\,w(x)\,dx$$

Idea of proof : use Hermite interpolation $p(x)$ w/ $p(x_i) = f(x_i)$, $p'(x_i) = f'(x_i)$, $i = 0, \dots, n$

7.4 Romberg integration

To approximate $\int_a^b f(x)\,dx$

- Composite trapezoid rule w/ n short intervals :

$$T(n) = \frac{b-a}{2n}\left(f(a) + 2f(a+2h) + \cdots + 2f(a+(n-1)\cdot 2h) + f(b)\right)$$

Denote $h = \frac{b-a}{2n}$

$$T(2n) = \frac{b-a}{4n}\Big(f(a) + 2f(a+h) + 2f(a+2h) + 2f(a+3h) + 2f(a+4h)$$

$$+ \cdots + 2f(a+(2n-2)h) + 2f(a+(2n-1)h) + f(b)\Big)$$

$$= \frac{1}{2}T(n) + h\sum_{i=1}^{n} f(a+(2i-1)h)$$

One can use this to calculate $T(1), T(2), T(4), \cdots$

$$T(1) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$T(2^n) = \frac{1}{2}T(2^{n-1}) + h\sum_{i=1}^{2^{n-1}} f(a+(2i-1)h), \quad h = \frac{b-a}{2^n}$$

- $T(n)$ has error term $a_2h^2 + a_4h^4 + \cdots$

One can use Richardson extrapolation to improve accuracy order

Denote $R(n,0) = T(2^n)$

$$\begin{array}{cccc}
R(0,0) & & & \\
R(1,0) & R(1,1) & & \\
R(2,0) & R(2,1) & R(2,2) & \ddots \\
\vdots & \vdots & \vdots & \\
R(M,0) & R(M,1) & R(M,2) & \cdots \\
\text{2nd} & \text{4th} & \text{6th} &
\end{array}$$

$$R(n,1) = \frac{1}{2^2-1}\left(2^2 R(n,0) - R(n-1,0)\right)$$ (this gives Simpson's)

$$R(n,2) = \frac{1}{2^4-1}\left(2^4 R(n,1) - R(n-1,1)\right)$$

$$R(n,m) = \frac{1}{2^{2m}-1}\left(2^{2m} R(n,m-1) - R(n-1,m-1)\right)$$

Then for fixed m, $R(n,m)$ has accuracy order $2m+2$.

- For fixed m and any $f \in C[a,b]$, we have $\lim_{n\to\infty} R(n,m) = \int_a^b f(x)\,dx$.

In fact, for $m=0$,

$$R(n,0) = \frac{1}{2}\left(h\sum_{i=0}^{2^n-1} f(a+ih) + h\sum_{i=1}^{2^n} f(a+ih)\right) \qquad h = \frac{b-a}{2^n}$$

Riemann sums

$\Rightarrow$ done for $m=0$.

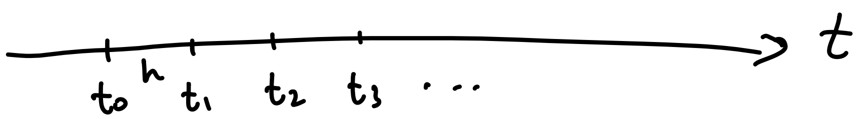
Then do induction on m.

Chapter 8 ODE solvers

Question: Given an ODE (initial value problems) for $x(t)$

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

how to approximate $x(t)$? (for $t \geq t_0$)



$$t_i = t_0 + ih, \quad i = 0, 1, 2, \cdots$$ h: spacing

Denote x_i as an approximation of $x(t_i)$

know: $x'(t_i) = f(t_i, x(t_i))$

$$\Rightarrow \quad \frac{x(t_{i+1}) - x(t_i)}{h} \approx f(t_i, x(t_i)) + O(h)$$

$$x(t_{i+1}) \approx x(t_i) + h f(t_i, x(t_i)) + O(h^2)$$

$$\Rightarrow \quad x_{i+1} = x_i + h f(t_i, x_i) \quad \underline{\text{(forward) Euler method}}$$
$$i \geq 0$$

<u>Error analysis</u>

Replace x_i by $x(t_i)$ in the method:

$$x(t_{i+1}) = x(t_i) + h f(t_i, x(t_i)) + \underbrace{O(h^2)}_{\text{local truncation error}}$$

- Even if x_i is the exact sol'n $x(t_i)$, x_{i+1} still differs from $x(t_{i+1})$ by $O(h^2)$

  $\rightsquigarrow$ L.T.E. can accumulate during iteration.

  $\rightarrow$ When solving ODE on a time interval of $O(1)$,

  $\#$ steps $= O(\frac{1}{h})$ "$\Rightarrow$" error $= O(h)$ (first order accurate).

<u>Thm</u> If $f(t,x)$ is Lipschitz in x and the exact sol'n $x(t) \in C^2$, then for fixed T, the numerical sol'n x_n by forward Euler w/ $h = \frac{T - t_0}{n}$ satisfies

$$|x(T) - x_n| \leq C(T) \cdot h.$$