Composite quadratures

One can divide a long interval $[a,b]$ into short ones and apply a quadrature rule on each

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$$

Take equally spaced pts

spacing $h = \dfrac{b-a}{N}$

$$x_i = a + ih, \quad i = 0, \cdots, N$$

Apply trapezoid rule on $[x_i, x_{i+1}]$ $i = 0, \cdots, N-1$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right)$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)\right)$$

Composite trapezoid rule.

$$\text{error} \le \sum_{i=0}^{N-1} \max_{x\in[x_i,x_{i+1}]} |f''(x)| \cdot \frac{1}{12} h^3$$

$$\le \max_{x\in[a,b]} |f''(x)| \cdot \frac{1}{12} h^3 N$$

$$= \max_{x\in[a,b]} |f''(x)| \cdot \frac{1}{12} h^2 (b-a)$$

$\hookrightarrow$ second order accuracy

Composite Simpson's rule (assume N is even)

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \frac{h}{3}\left(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right)$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \frac{h}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4)$$

$$+ \cdots + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)\Big)$$

$$\text{error} \le \max_{x\in[a,b]} |f^{(4)}(x)| \cdot \frac{1}{90} h^5 \cdot \frac{N}{2}$$

$$= \max_{x\in[a,b]} |f^{(4)}(x)| \cdot \frac{1}{180} h^4 (b-a)$$

↳ 4-th order accuracy

7.3 Gaussian quadrature

Idea : for a quadrature

$$\underbrace{\int_a^b f(x)\, w(x)\, dx \approx \sum_{i=0}^{n} A_i f(x_i)}\ (*)$$

($w(x) > 0$ is a weight function)

try to choose $\underbrace{A_0, \cdots, A_n, x_0, \cdots, x_n}_{2n+2}$ to make it exact for

deg $\le \underline{2n+1}$ polynomials

$$\int_{-1}^{1} x^2\, dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$$

Ex $[a,b] = [-1, 1]$, $w(x) = 1$, $n=1$

$f(x) = 1$: $\quad 2 = A_0 \cdot 1 + A_1 \cdot 1$

$f(x) = x$: $\quad 0 = A_0 x_0 + A_1 x_1$

$f(x) = x^2$: $\quad \frac{2}{3} = A_0 x_0^2 + A_1 x_1^2$

$f(x) = x^3$: $\quad 0 = A_0 x_0^3 + A_1 x_1^3$

Assume $x_1 = -x_0$, $A_1 = A_0$

then $A_0 = 1$, $\frac{2}{3} = 2 x_0^2$

$x_0^2 = \frac{1}{3}$ $\quad x_0 = \pm\frac{1}{\sqrt{3}}$ $\quad$ take $x_0 = -\frac{1}{\sqrt{3}}$

$$\int_{-1}^{1} f(x)\, dx \approx 1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right)$$

Thm Let q be a nonzero polynomial of degree $n+1$ that is
<u>w-orthogonal</u> to Π_n (space of poly. w/ deg $\le n$), i.e.

$$\int_a^b q(x)\, f(x)\, w(x)\,dx = 0 \qquad \forall f \in \Pi_n$$

Then
- q has $n+1$ distinct zeros in (a,b) (denoted $x_0, \cdots, x_n$)
- By choosing A_i properly, the quadrature $(*)$ is exact for poly. w/ deg $\le 2n+1$

(this quadrature is called <u>Gaussian quadrature</u>)

Proof ① We claim that q changes sign at least $n+1$ times on $[a,b]$

Suppose not, then q changes sign r times w/ $\underset{\substack{\uparrow \\ \text{by orthogonality} \\ \text{against } f\equiv 1.}}{1 \le r} \le n$

$\exists\ a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b$

so that q has one sign in each (t_i, t_{i+1})

Consider $p(x) = \prod_{i=1}^{r} (x - t_i) \in \Pi_n$. Then $p \cdot q$ has one sign on $[a,b]$

$\Rightarrow \int_a^b p(x)\, q(x)\, w(x)\,dx \ne 0$ contradiction to orthogonality between p and q

The claim implies that q has at least $n+1$ distinct zeros in (a,b)

Since $\deg(q) = n+1$, q has $n+1$ distinct simple zeros in (a,b)

② Choose A_i so that $(*)$ is exact for deg $\le n$ poly.

For $f \in \Pi_n$, $\int_a^b f(x)\, w(x)\,dx = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x)\, w(x)\,dx$

$$= \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)\, w(x)\,dx}_{A_i}$$

③ Prove that (*) is exact for any $f \in \Pi_{2n+1}$

$$f = q\,p + r \quad \text{where} \quad \deg(p) \le n\ ,\ \deg(r) \le n$$

$\deg \le 2n+1$ $\quad$ $\deg = n+1$

$$f(x_i) = \underbrace{q(x_i)}_{=0}\, p(x_i) + r(x_i) = r(x_i)$$

$$\int_a^b f(x)\,w(x)\,dx = \int_a^b \big(q(x)\,p(x) + r(x)\big)w(x)\,dx = \int_a^b r(x)\,w(x)\,dx$$

$\int_a^b q \cdot p\ w\,dx = 0$ by orthogonality

$$= \sum_{i=0}^{\hat{n}} A_i\, r(x_i)$$

by ②

$$= \sum_{i=0}^{\hat{n}} A_i\, f(x_i)$$

- The polynomial q_{n+1} for various n can be obtained by Gram-Schmidt on $\{1, x, x^2, \cdots\}$ $\quad$ "orthogonal polynomials" $q_0, q_1, \cdots$

<u>Ex</u> $\quad [a,b] = [-1,1]\ ,\ w(x) = 1$

$$q_0 = 1$$

$$q_1 = x - \frac{\int_{-1}^{1} x \cdot q_0\,dx}{\int_{-1}^{1} q_0 \cdot q_0\,dx}\, q_0 = x$$

$$q_2 = x^2 - \frac{\int_{-1}^{1} x^2 \cdot q_0\,dx}{\int_{-1}^{1} q_0 \cdot q_0\,dx}\, q_0 - \frac{\int_{-1}^{1} x^2 \cdot q_1\,dx}{\int_{-1}^{1} q_1 \cdot q_1\,dx}\, q_1 = x^2 - \frac{1}{3}$$

for $n = 1$, the Gaussian quadrature nodes are $\pm \frac{1}{\sqrt{3}}$

"Legendre polynomials"