Problem 1: The Gauss-Lobatto quadrature on the interval $[-1,1]$ is a quadrature rule of the form

$$\int_{-1}^{1} f(x)dx \approx A_0 f(-1) + A_n f(1) + \sum_{i=1}^{n-1} A_i f(x_i)$$

where $x_1,\dots,x_{n-1},A_0,\dots,A_n$ are chosen so that the quadrature is exact for any polynomial with degree no more than $2n-1$. The following procedure gives the construction of the Gauss-Lobatto quadrature.

(1) Let $P_n(x)$ denote the Legendre polynomial of degree n (i.e., orthogonal polynomial on $[-1,1]$ with weight function $w(x)=1$). Prove that $P_n'(x)$ has $n-1$ distinct zeros in $(-1,1)$.

(2) Denote the zeros of $P_n'(x)$ as $x_1,\dots,x_{n-1}$. Prove that the above quadrature is exact for any polynomial with degree no more than n if $A_0,\dots,A_n$ are properly chosen.

(3) For any polynomial f with degree no more than $2n-1$, do the polynomial division

$$f(x) = p(x)q(x) + r(x), \quad \deg(p) \le n-2,\ \deg(r) \le n$$

where $q(x) := (x-1)(x+1)P_n'(x)$ is a polynomial of degree $n+1$. Use this to prove that the quadrature is exact for f.

Problem 2: Prove that the weights $A_0,\dots,A_n$ in Problem 1 are positive.

Problem 3: Derive the order conditions up to second order for a general explicit Runge-Kutta method.

Problem 4: A diagonally implicit RK method is given by a Butcher table whose $\{a_{jk}\}$ matrix is lower-triangular. Each stage is given by the implicit equation

$$x^{(j)} = x_i + h\sum_{k=1}^{j} a_{jk} f(t_i + c_k h, x^{(k)}), \quad j = 1,\dots,s$$

Consider a diagonally implicit RK method with Butcher table

$$\begin{array}{c|cc} c_1 & \gamma & 0 \\ c_2 & \delta & \gamma \\ \hline & \delta & \gamma \end{array}$$

Determine δ and γ so that this method has second order accuracy.