Error estimate of Hermite interpolation, case $k_i = 2$

<u>Thm</u> Let $x_0, \cdots, x_n$ be distinct pts in $[a,b]$, $f \in C^{2n+2}[a,b]$.

If p is the Hermite interpolation of f at $x_0, \cdots, x_n$ w/ $k_i = 2$, $i = 0, \cdots, n$

then

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2$$

<u>Proof</u> Without loss of generality, assume x is not any x_i.

Define $w(t) = \prod_{i=0}^{n} (t - x_i)^2$, $\phi(t) = f(t) - p(t) - \lambda w(t)$

$\lambda = \dfrac{f(x) - p(x)}{w(x)}$ makes $\phi(x) = 0$.

$\phi = 0$ at $\underbrace{x_0, \cdots, x_n, x}_{n+2 \text{ pts}}$

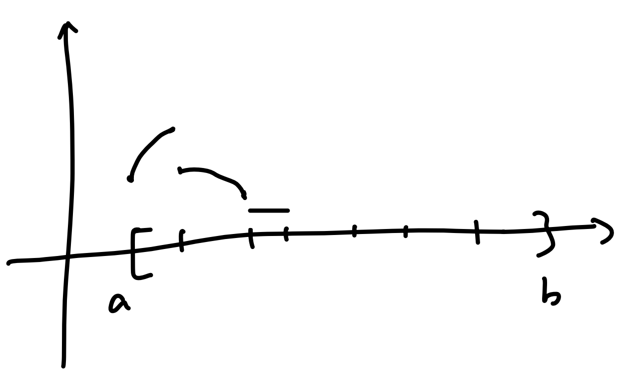
$\xRightarrow{\text{Rolle's}}$ $\phi' = 0$ at $n+1$ pts distinct from $x_0, \cdots, x_n, x$

$\phi'(t) = f'(t) - p'(t) - \lambda w'(t) = 0$ at $x_0, \cdots, x_n$

$\Rightarrow \phi' = 0$ at $2n+2$ pts

$\Rightarrow \cdots \Rightarrow \phi^{(2n+2)} = 0$ at 1 pt, call it ξ_x ...

6.4 Spline interpolation



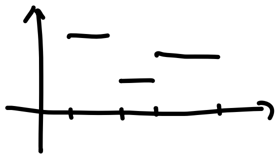
Idea: approximate a function by piecewise polynomials w/ continuity conditions at endpts.

Def Let $t_0 < t_1 < \cdots < t_n$ be the knots. A spline function of degree k is a function S s.t.

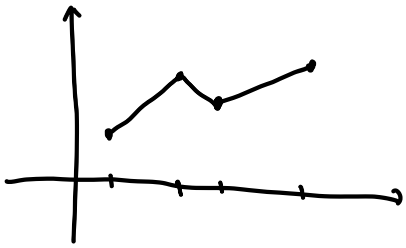
(1) on each $[t_{i-1}, t_i)$, S is a polynomial of degree $\le k$

(2) S has a continuous $(k-1)$-th derivative on $[t_0, t_n]$

$k=0$: piecewise constant

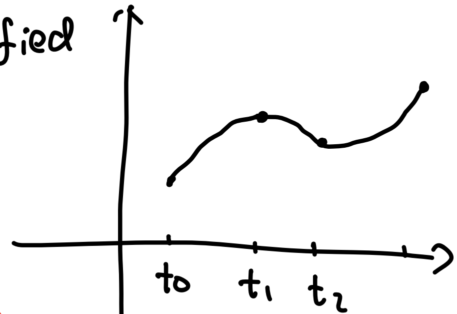


$k=1$: piecewise linear continuous function



Cubic spline ($k=3$)

Suppose $S(t_i) = y_i$, $i = 0, \cdots, n$ are specified



Let S_i be the cubic poly. on $[t_i, t_{i+1}]$

$i = 0, \cdots, n-1$

\# of unknowns $= 4n$

Then

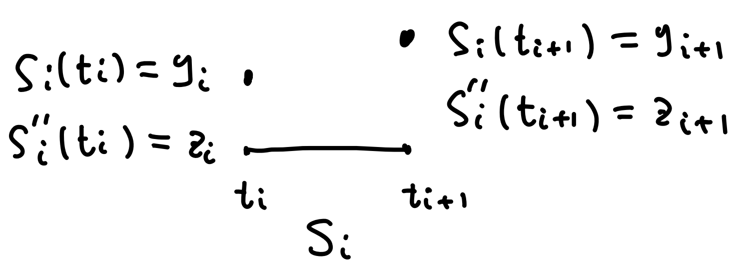
$$S_i(t_i) = y_i \quad , \quad S_i(t_{i+1}) = y_{i+1}$$

At each t_i, $i = 1, \cdots, n-1$, we need

$$S_{i-1}'(t_i) = S_i'(t_i) \quad , \quad S_{i-1}''(t_i) = S_i''(t_i)$$

\# of Conditions $= 2n + 2(n-1) = 4n-2$

Denote $z_i = S''(t_i)$, $i = 0, \cdots, n$.



$$S_i''(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i) \qquad h_i := t_{i+1} - t_i$$

$$\Rightarrow \quad S_i(x) = \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + \text{linear poly.}$$

$$\left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1} - x) + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}\right)(x - t_i)$$

$$S_i'(x) = -\frac{z_i}{2h_i}(t_{i+1} - x)^2 + \frac{z_{i+1}}{2h_i}(x - t_i)^2 - \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right) + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}\right)$$

$$S_i'(t_i) = -\frac{z_i h_i}{2} - \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right) + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}\right)$$

$$= \boxed{-\frac{z_i h_i}{3} - \frac{y_i}{h_i} + \frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}}$$

$$S_i'(t_{i+1}) = \frac{z_{i+1} h_i}{2} - \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right) + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}\right)$$

$$= \frac{z_{i+1} h_i}{3} - \frac{y_i}{h_i} + \frac{z_i h_i}{6} + \frac{y_{i+1}}{h_i}$$

$$S_{i-1}'(t_i) = \boxed{\frac{z_i h_{i-1}}{3} - \frac{y_{i-1}}{h_{i-1}} + \frac{z_{i-1} h_{i-1}}{6} + \frac{y_i}{h_{i-1}}}$$

$$=$$

$$\frac{h_{i-1}}{6} z_{i-1} + \frac{h_i + h_{i-1}}{3} z_i + \frac{h_i}{6} z_{i+1} = -\frac{y_i}{h_i} + \frac{y_{i+1}}{h_i} + \frac{y_{i-1}}{h_{i-1}} - \frac{y_i}{h_{i-1}}$$

$$h_{i-1} z_{i-1} + \underbrace{2(h_i + h_{i-1})}_{u_i} z_i + h_i z_{i+1} = \underbrace{6\left(-\frac{y_i}{h_i} + \frac{y_{i+1}}{h_i} + \frac{y_{i-1}}{h_{i-1}} - \frac{y_i}{h_{i-1}}\right)}_{v_i} \qquad i = 1, \ldots, n-1$$

Impose two extra conditions : $z_0 = z_n = 0$. "natural cubic spline"

$$\begin{pmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-3} & u_{n-2} & h_{n-2} \\ & & & & h_{n-2} & u_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{pmatrix}$$

This matrix is tri-diagonal, symmetric, diagonally dominant.

One can solve it by Gaussian elimination w/o pivoting w/ cost $O(n)$.

- Once $z_1, \cdots, z_{n-1}$ are obtained, to evaluate $S(x)$ at given x:

  (1) find which interval contains x:

  $$(-\infty, t_1), [t_1, t_2), \cdots, [t_{n-2}, t_{n-1}), [t_{n-1}, \infty)$$

  (by bisection, cost is $O(\log n)$ comparisons)

  (2) Apply formula for $S_i(x)$ ($O(1)$ multiplication/division).

<u>Thm</u> If $f \in C^2[a,b]$ and $a = t_0 < t_1 < \cdots < t_n = b$, then the natural cubic spline S for f satisfies

$$\int_a^b (S''(x))^2 \, dx \le \int_a^b (f''(x))^2 \, dx$$

- In other words, if $E[\phi] = \int_a^b (\phi''(x))^2 dx$, then among all C^2 functions ϕ w/ $\phi(t_i) = f(t_i)$, $i = 0, \cdots, n$, $\phi = S$ achieves the minimum of E.

"elastic bending energy"

$$\phi''(x) \approx \frac{\phi''(x)}{(1 + (\phi'(x))^2)^{3/2}} \quad \text{curvature}$$

if ϕ' is small

<u>Proof</u> Let $g = f - S$

$$\int_a^b (f'')^2 dx = \int_a^b (S'' + g'')^2 dx$$

$$= \int_a^b (S'')^2 dx + \underbrace{\int_a^b (g'')^2 dx}_{\ge 0} + 2 \underbrace{\int_a^b S'' g'' dx}_{\text{can be proved } = 0}$$