Thm If $f \in C^n[a,b]$ and $x_0, \cdots, x_n$ are distinct pts in $[a,b]$

then $$f[x_0, \cdots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

for some $\xi \in (a,b)$.

Proof Let p be the poly. interp. of f at $x_0, \cdots, x_{n-1}$.

$\hookrightarrow$ deg $\leq n-1$

Apply error estimate at x_n for p,

$$f(x_n) - p(x_n) = \frac{1}{n!} f^{(n)}(\xi) \prod_{j=0}^{n-1} (x_n - x_j)$$

Claim: $$f(x_n) - p(x_n) = f[x_0, \cdots, x_n] \prod_{j=0}^{n-1} (x_n - x_j)$$

In fact, let q be the poly. interp. at $x_0, \cdots, x_n$.

Then by Newton form,

$$q(x) = p(x) + f[x_0, \cdots, x_n](x-x_0)\cdots(x-x_{n-1})$$

Evaluate at $x = x_n$,

$$q(x_n) = p(x_n) + f[x_0, \cdots, x_n](x_n - x_0)\cdots(x_n - x_{n-1})$$

$\hookrightarrow = f(x_n)$

and we get the claim.

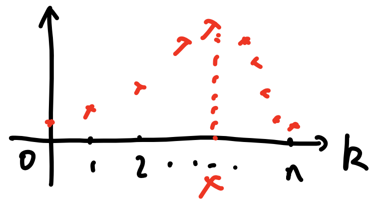
- For a general $f \in C[a,b]$, to get a polynomial approximation which converges to f uniformly as degree $\to \infty$, one way is the Bernstein polynomials (for $[0,1]$)

$$B_n f = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g_{nk}(x) \quad \text{where} \quad g_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{g_{nk}(x)}$$

for fixed x



One can prove $\|B_n f - f\|_{L^\infty([0,1])} \to 0$ as $n \to \infty$.

6.3 Hermite interpolation

Question : Given a function f and nodes $x_0, \cdots, x_n$, want to find a polynomial p s.t. it agrees w/ f at $x_0, \cdots, x_n$ together w/ agreement of some derivatives.

To be precise, given distinct nodes $x_0, \cdots, x_n$ and positive integers $k_0, \cdots, k_n$, we want a polynomial p of degree $\le k_0 + k_1 + \cdots + k_n - 1 := m$ s.t.

$$p^{(j)}(x_i) = f^{(j)}(x_i) \qquad 0 \le i \le n, \ 0 \le j \le k_i - 1$$

called the Hermite interpolation

Thm There exists a unique Hermite interpolation.

Proof Suppose $p(x) = \sum_{\ell=0}^{m} a_\ell x^\ell$. Then the conditions are

$$\sum_{\ell=j}^{m} a_\ell \cdot \ell(\ell-1)\cdots(\ell-j+1) x_i^{\ell-j} = f^{(j)}(x_i)$$

$$0 \le i \le n, \ 0 \le j \le k_i - 1$$

$$p^{(j)}(x) = \sum_{\ell=j}^{m} a_\ell \, \ell(\ell-1)\cdots(\ell-j+1) x^{\ell-j}$$

These are $m+1$ linear conditions in $m+1$ unknowns $a_0, \cdots, a_m$.

Therefore, this problem always has a unique sol'n $\quad A\vec{x} = \vec{b}$

$\iff$ the corresponding homogeneous problem $\quad A\vec{x} = \vec{0}$

only has the trivial sol'n

The homogeneous problem : find poly. p of $\deg \le m$ s.t.

$$p^{(j)}(x_i) = 0 \qquad 0 \le i \le n,\ 0 \le j \le k_i - 1$$

For each i, $p(x)$ has to contain a factor $(x - x_i)^{k_i}$

Therefore, $p(x)$ is a multiple of $\underbrace{\prod_{i=0}^{n} (x - x_i)^{k_i}}_{\deg = k_0 + \cdots + k_n = m+1}$

$\Rightarrow p = 0$.

Divided differences for Hermite interpolation

- Given $f(x_i)$, $f'(x_i)$ is almost the same as given $f(x_i)$, $f(x_i + \varepsilon)$

  because $f'(x_i) \approx \dfrac{f(x_i + \varepsilon) - f(x_i)}{\varepsilon}$

- This motivates to consider repeated nodes in divided differences

For example, given $f(x_0)$, $f'(x_0)$, $f(x_1)$

$$f[x_0, x_0 + \varepsilon] = \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}$$

$$\begin{array}{llll}
x_0 & f[x_0] & & \\
 & & f[x_0, x_0] := f'(x_0) & \\
x_0 & f[x_0] & & f[x_0, x_0, x_1] \\
 & & f[x_0, x_1] & \\
x_1 & f[x_1] & &
\end{array}$$

○ : known

$$p(x) = f[x_0] + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)(x - x_0)$$

Given $f(x_0)$, $f(x_1)$, $f'(x_1)$, $f''(x_1)$

$$
\begin{array}{lllll}
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & f[x_0, x_1, x_1] & \\
 & & f[x_1, x_1] := f'(x_1) & & f[x_0, x_1, x_1, x_1] \\
x_1 & f[x_1] & & f[x_1, x_1, x_1] := \frac{1}{2!} f''(x_1) & \\
 & & f[x_1, x_1] := f'(x_1) & & \\
x_1 & f[x_1] & & &
\end{array}
$$

Lagrange form

Suppose we can find $\deg \le m$ polynomial l_{ij}, $0 \le i \le n$, $0 \le j \le k_i$

s.t.
$$l_{ij}^{(j')}(x_{i'}) = \begin{cases} 1 & (i,j) = (i', j') \\ 0 & \text{otherwise} \end{cases} \qquad 0 \le i' \le n,\ 0 \le j' \le k_{i'}$$

then we have

$$p(x) = \sum_{\substack{0 \le i \le n \\ 0 \le j \le k_i - 1}} f^{(j)}(x_i)\, l_{ij}(x)$$

Construction in the case $k_i = 2$ $\forall\, 0 \le i \le n$.

$$l_{i0}(x) = \left(1 - 2(x - x_i)\, l_i'(x_i)\right) l_i^2(x) \qquad \deg l_{i0} = 2n+1 = m$$

$$l_{i1}(x) = (x - x_i)\, l_i^2(x) \qquad l_i(x_{i'}) = \delta_{ii'}$$

where $l_i(x)$ is the Lagrange basis function in poly. interp.

Check l_{i0} property:

$$l_{i0}(x_i) = \left(1 - \underline{2\cancel{(x_i - x_i)\, l_i'(x_i)}}\right) l_i^2(x_i) = 1$$

$$l_{i0}(x_{i'}) = \left(1 - 2(x_{i'} - x_i)\, l_i'(x_i)\right) l_i^2(x_{i'}) = 0$$
$(i' \ne i)$

$$l_{i0}'(x) = -2\, l_i'(x_i)\, l_i^2(x) + \left(1 - 2(x - x_i)\, l_i'(x_i)\right) \cdot 2\, l_i(x)\, l_i'(x)$$

$$l_{i0}'(x_i) = -2\, l_i'(x_i)\, \cancel{l_i^2(x_i)} + \left(1 - \underline{2\cancel{(x_i - x_i)\, l_i'(x_i)}}\right) \cdot 2\, \cancel{l_i(x_i)}\, l_i'(x_i)$$
$$= 0$$

$$l_{i0}'(x_{i'}) = -2\, l_i'(x_i)\, \underbrace{l_i^2(x_{i'})}_{=0} + \left(1-2(x_{i'}-x_i)\, l_i'(x_i)\right)\cdot 2\, \underbrace{l_i(x_{i'})}_{=0}\, l_i'(x_{i'})$$

$(i' \neq i)$

$$= 0 .$$