Error analysis of polynomial interpolation

<u>Thm</u> Let $f \in C^{n+1}[a,b]$, and p be the poly. interp. of f at distinct pts $x_0, \cdots, x_n \in [a,b]$. Then, for $x \in (a,b)$

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

for some $\xi_x \in (a,b)$.

<u>Proof</u> Assume x is not any x_i. Let

$$w(t) = \prod_{i=0}^{n} (t - x_i), \quad \phi(t) = f(t) - p(t) - \lambda w(t)$$

$$\lambda = \frac{f(x) - p(x)}{w(x)} \quad \text{(which makes } \phi(x) = 0\text{)}$$

$$\phi = 0 \quad \text{at} \quad \underbrace{x, x_0, \cdots, x_n}_{n+2 \text{ distinct pts}}$$

$$\overset{\text{Rolle's Thm}}{\Longrightarrow} \phi' = 0 \quad \text{at} \quad n+1 \text{ pts}$$

$$\Rightarrow \phi'' = 0 \quad \text{at} \quad n \text{ pts}$$

$$\vdots$$

$$\Rightarrow \phi^{(n+1)} = 0 \quad \text{at} \quad 1 \text{ pt, call it } \xi_x \in (a,b)$$

$$\phi^{(n+1)}(t) = f^{(n+1)}(t) - \lambda (n+1)!$$

evaluate at $t = \xi_x$

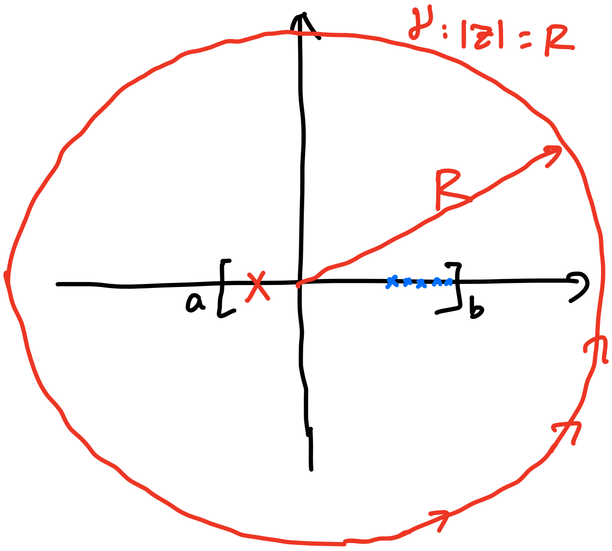
$$0 = f^{(n+1)}(\xi_x) - \frac{f(x) - p(x)}{w(x)} (n+1)!$$

- Generally speaking, poly. interp. of a continuous function may not converge to itself, even for C^∞ functions

  ($f^{(n+1)}$ may increase very fast as n increases)

- Poly. interp. converges for certain holomorphic functions.

  Suppose $f(z)$ is an entire function. Then



$$f(x) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-x}\, dz$$

$$f^{(n+1)}(x) = \frac{1}{2\pi i} (n+1)! \oint_\gamma \frac{f(z)}{(z-x)^{n+2}}\, dz$$

$$x \in [a,b] \quad , \quad R \geq 2 \max\{|a|, |b|\}$$

$$|f^{(n+1)}(x)| \leq \frac{1}{\cancel{2\pi i}} (n+1)! \cdot \cancel{2\pi} R \cdot \max_{|z|=R} |f(z)| \cdot \frac{1}{(R/2)^{n+2}}$$

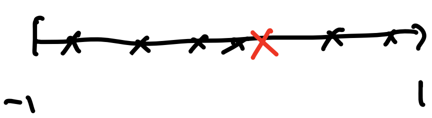
$$= (n+1)!\ R^{-(n+1)} \cdot 2^{n+2} \cdot \max_{|z|=R} |f(z)|$$

$$|f(x) - p(x)| \leq \cancel{(n+1)!}\ R^{-(n+1)} \cdot 2^{n+2} \cdot \max_{|z|=R} |f(z)| \cdot \frac{1}{\cancel{(n+1)!}} \cdot (b-a)^{n+1}$$

$$\to 0 \quad \text{as } n \to \infty$$

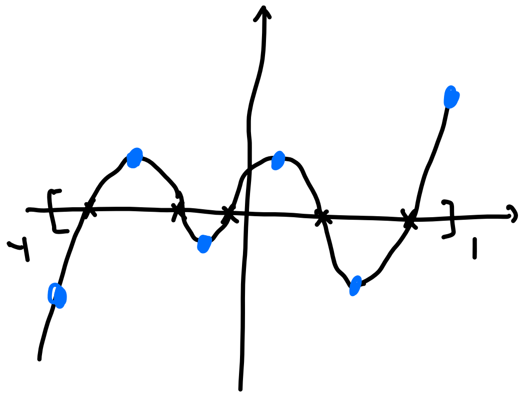
$$\text{if} \quad R > 2(b-a)$$

- Improvement on the term $\prod_{i=0}^{n} (x - x_i)$ (say, on $[-1, 1]$)



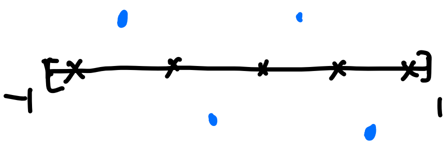
Want to choose $\{x_0, \cdots, x_n\}$ to minimize

$$\max_{x\in[-1,1]} \left| \prod_{i=0}^{n} (x-x_i) \right|$$



intuitively, we need all the local max/mins to have same absolute values.

Uniform distribution of nodes is not expected to be optimal. One needs less spacing when near the endpts.



$\prod_{i=0}^{n}(x-x_i)$

Thm If p is monic of degree n, then $\max_{x\in[-1,1]} |p(x)| \geq 2^{1-n}$.

↳ leading coeff. $= 1$

"=" is achieved for the Chebyshev polynomial

$$T_n(x) = \cos(n \cos^{-1} x)$$

(after normalization)

- $T_n(x)$ is a deg-n poly. In fact,

$$\cos((n+1)\theta) = \cos(n\theta + \theta) = \cos n\theta \cdot \cos\theta - \sin n\theta \sin\theta$$

$$\cos((n-1)\theta) = \cos(n\theta - \theta) = \cos n\theta \cos\theta + \sin n\theta \sin\theta$$

$$\Rightarrow \cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos n\theta \cos\theta$$

Take $\theta = \cos^{-1} x$

$$T_{n+1}(x) + T_{n-1}(x) = 2x T_n(x)$$

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$

$T_0(x) = 1$, $T_1(x) = x$

By induction, $T_n(x)$ is a poly. of deg n $(n \geq 0)$

the leading coeff. of $T_n(x)$ is 2^{n-1} $(n \geq 1)$

- $\max_{x \in [-1,1]} |T_n(x)| = 1$

$\Rightarrow$ the monic poly. $\frac{1}{2^{n-1}} T_n(x)$ satisfies

$$\max_{x \in [-1,1]} \left| \frac{1}{2^{n-1}} T_n(x) \right| = 2^{1-n}$$

Proof of "other $p(x)$ are worse"

Suppose $p(x)$ monic, deg-n, w/ $|p(x)| < 2^{1-n}$ $\forall\ -1 \leq x \leq 1$

$q(x) = 2^{1-n} T_n(x)$, $a_k = \cos \frac{k\pi}{n}$ $k = 0, \cdots, n$

$\cos(n \cos^{-1} x)$ $\quad n \cos^{-1} x = k\pi \quad x = \cos \frac{k\pi}{n}$

$q(a_k) = 2^{1-n} \cos(n \cdot \frac{k\pi}{n}) = 2^{1-n} (-1)^k$

$(-1)^k (q(a_k) - p(a_k)) > 0$ $\quad k = 0, \cdots, n$

$q - p$ $\quad \cdots \quad + \quad - \quad +$

$\Rightarrow$ $q - p$ has a root in each (a_{k+1}, a_k) , $k = 0, \cdots, n-1$

(n roots in total)

This is impossible bc $q - p$ has degree $\leq n - 1$. Contradiction !

- The zeros of $T_{n+1}(x)$ are optimal choice of nodes in view of the error estimate.

$$\cos((n+1)\cos^{-1} x) = 0 \qquad (n+1)\cos^{-1} x = (k + \tfrac{1}{2})\pi$$

$$x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right) \qquad k = 0, \cdots, n$$

In this case, error estimate :

$$|f(x) - p(x)| \leq \frac{1}{(n+1)!} \max_{t\in[-1,1]} \left| f^{(n+1)}(t) \right| \cdot 2^{-n} . \qquad \forall\, x \in [-1,1]$$