Problem 1: Formulate a statement about the uniqueness of QR-factorization for invertible square matrices and prove it. (imitate that for LU-factorization)

Problem 2: Prove the following error estimate for the power method for eigenvalues: Let A be an $n \times n$ diagonalizable complex matrix, with eigenvalues $\lambda_1, \dots, \lambda_n$ satisfying

$$|\lambda_j| \le \beta |\lambda_1|, \quad j = 2, \dots, n$$

for some $0 < \beta < 1$. Denote $\mathbf{v}_1, \dots, \mathbf{v}_n$ as the corresponding eigenvectors. Let the initial vector

$$\mathbf{x}_0 = \sum_{j=1}^{n} c_j \mathbf{v}_j$$

with $c_1 \neq 0$. Let ϕ be a linear functional on $\mathbb{C}^n$ satisfying $\phi(\mathbf{v}_1) \neq 0$. Prove that the k-th iteration $\mathbf{x}_k$ satisfies

$$\left| \frac{\phi(\mathbf{x}_{k+1})}{\phi(\mathbf{x}_k)} - \lambda_1 \right| \le C\beta^k$$

for suitable constant $C > 0$. (here you also need to prove that $\phi(\mathbf{x}_k) \neq 0$ for sufficiently large k so that the above fraction is well-defined.)

Problem 3: Consider the distinct nodes $x_0, \dots, x_n$ in $[a, b]$, and a function f. Let $p_{i,j}$ denote the polynomial interpolation of f at $x_i, x_{i+1}, \dots, x_j$ for any $0 \le i \le j \le n$. In the lecture we proved

$$p_{0,n}(x) = p_{0,n-1}(x) \cdot \frac{x_n - x}{x_n - x_0} + p_{1,n}(x) \cdot \frac{x - x_0}{x_n - x_0}$$

(in the proof of iterative formula for divided difference, with a slightly different notation).

(1) Use it to prove the following generalization: for any $1 \le k \le n$, there exists polynomials $q_{0,n-k}, \dots, q_{k,n}$ of degree at most k, such that

$$p_{0,n}(x) = \sum_{j=0}^{k} p_{j,n-k+j}(x) q_{j,n-k+j}(x)$$

(2) Derive the explicit formulas for the q polynomials when $k = 2$.

(3) What are the q polynomials when $k = n$?

Problem 4: Consider the following variant of the Hermite interpolation: take the distinct nodes $x_0, \dots, x_n$ in $[a, b]$, a positive integer K and a function f (with enough regularity). We want to find a polynomial p of degree at most $m = (n+1)K$ such that the following $m+1$ conditions are satisfied:

$$p^{(j)}(x_i) = f^{(j)}(x_i), \quad 0 \le i \le n,\ 0 \le j \le K-1; \qquad \int_a^b p(x)dx = \int_a^b f(x)dx$$

Determine whether this problem always have a solution for any $n, a, b, x_0, \dots, x_n, f$, for the following choices of K: if yes, prove it; if no, find an explicit example so that the solution does not exist or is not unique.

(1) $K = 2$;

(2) $K = 3$.