4.6 (continued)

$$\vec{x}^{(k)} = G\vec{x}^{(k-1)} + \vec{c} \quad \text{converges}$$

$\Leftrightarrow$ you can find sub. matrix norm s.t. $\|G\| < 1$

$\Leftrightarrow$ $\rho(G) < 1$

Thm If A is row or column diagonally dominant, then Gauss-Seidel method converges.

$$Q\vec{x} = (Q - A)\vec{x} + \vec{b} \qquad Q = \text{lower-triangular part of } A$$

$$G = I - Q^{-1}A.$$

proof It suffices to show $\rho(I - Q^{-1}A) < 1$.

Let λ be an eigenvalue of $I - Q^{-1}A$, w/ eigenvector $\vec{x} \neq \vec{0}$

$$(I - Q^{-1}A)\vec{x} = \lambda\vec{x}$$

$$\vec{x} - Q^{-1}A\vec{x} = \lambda\vec{x}$$

$$Q^{-1}A\vec{x} = \vec{x} - \lambda\vec{x}$$

$$A\vec{x} = Q\vec{x} - \lambda Q\vec{x}$$

$$\underbrace{(Q - A)}_{\substack{-\text{ str. upper-tri.} \\ \text{part of } A}}\vec{x} = \lambda Q\vec{x}$$

$$-\sum_{j=i+1}^{n} a_{ij}x_j = \lambda\sum_{j=1}^{i-1} a_{ij}x_j + \lambda a_{ii}x_i. \qquad i = 1, \cdots, n$$

Suppose $|\lambda| \geq 1$. Take i such that $|x_i|$ is largest.

Then $|\lambda a_{ii}x_i| = |\lambda| \cdot |a_{ii}| \cdot |x_i|$

$$\left| -\sum_{j=i+1}^{n} a_{ij} x_j - \lambda \sum_{j=1}^{i-1} a_{ij} x_j \right|$$

$$\le \sum_{j=i+1}^{n} |a_{ij}|\cdot|x_j| + |\lambda| \sum_{j=1}^{i-1} |a_{ij}|\cdot|x_j|$$

$$\le |\lambda| \sum_{j=i+1}^{n} |a_{ij}|\cdot|x_i| + |\lambda| \sum_{j=1}^{i-1} |a_{ij}|\cdot|x_i|$$

$$= |\lambda|\cdot|x_i|\cdot \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| < |\lambda|\cdot|x_i|\cdot|a_{ii}|$$ Contradiction!

This gives $|\lambda| < 1$ $\Rightarrow$ $\rho(I - Q^{-1}A) < 1$.

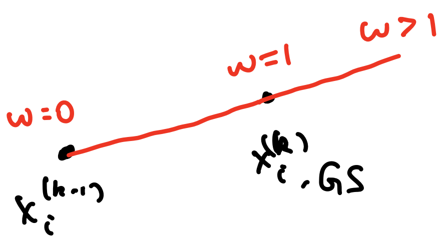
Successive over-relaxation method (SOR)

Gauss-Seidel : $$x_i^{(k)} = \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right)$$

$$\Downarrow$$

SOR : $$x_i^{(k)} = (1-\omega) x_i^{(k-1)} + \omega \frac{1}{a_{ii}} \left(-\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i \right)$$

$0 < \omega < 2$



$$Q \vec{x}^{(k)} = (Q - A) \vec{x}^{(k-1)} + \vec{b}$$

$$A = D - C_L - C_u$$

$$\vec{x}^{(k)} = (1-\omega) \vec{x}^{(k-1)} + \omega D^{-1} \left(C_L \vec{x}^{(k)} + C_u \vec{x}^{(k-1)} + \vec{b} \right)$$

$$\omega^{-1} D \vec{x}^{(k)} = \omega^{-1}(1-\omega) D \vec{x}^{(k-1)} + C_L \vec{x}^{(k)} + C_u \vec{x}^{(k-1)} + \vec{b}$$

$$(\omega^{-1} D - C_L) \vec{x}^{(k)} = \left(\underbrace{\omega^{-1}(1-\omega) D + C_u} \right) \vec{x}^{(k-1)} + \vec{b}$$

$$\uparrow$$
$$Q = \omega^{-1} D - C_L \qquad Q-A$$

<u>Thm</u> If A is symmetric and positive definite, and $0 < \omega < 2$, then SOR converges.

<u>Proof</u> It suffices to prove $\rho(G) < 1$, where

$$G = I - Q^{-1}A = I - (\omega^{-1}D - C_L)^{-1}A$$

Let λ be an eigenvalue of G, w/ eigenvector $\vec{x}$

$$(I - (\omega^{-1}D - C_L)^{-1}A)\vec{x} = \lambda\vec{x} \qquad A = D - C_L - C_L^T$$

$$\vec{x} - (\omega^{-1}D - C_L)^{-1}A\vec{x} = \lambda\vec{x}$$

$$(\omega^{-1}D - C_L)^{-1}A\vec{x} = (1-\lambda)\vec{x}$$

$\vec{x}^*$: transpose conjugate of $\vec{x}$

$$A\vec{x} = (1-\lambda)(\omega^{-1}D - C_L)\vec{x}$$

$$\vec{x}^* A\vec{x} = (1-\lambda)\,\vec{x}^*(\omega^{-1}D - C_L)\vec{x}$$

$$\overline{\vec{x}^* C_L \vec{x}} = (\vec{x}^* C_L \vec{x})^* = \vec{x}^* C_L^T \vec{x}$$

Suppose $\lambda \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$

$$\vec{x}^T A\vec{x} = (1-\lambda)\,\vec{x}^T(\omega^{-1}D - \tfrac{1}{2}C_L - \tfrac{1}{2}C_L^T)\vec{x}$$

$$= (1-\lambda)\,\vec{x}^T((\omega^{-1} - \tfrac{1}{2})D + \tfrac{1}{2}A)\vec{x}$$

$$(\underbrace{1 - \tfrac{1}{2}(1-\lambda)}_{=\frac{1}{2}(1+\lambda)})\underbrace{\vec{x}^T A\vec{x}}_{>0} = (1-\lambda)\underbrace{(\omega^{-1} - \tfrac{1}{2})}_{>0}\underbrace{\vec{x}^T D\vec{x}}_{>0}$$

$$\Rightarrow (1+\lambda)(1-\lambda) > 0 \quad \Rightarrow -1 < \lambda < 1$$

- When A is sym. posdef. and tri-diagonal, the optimal ω is

$$\omega_0 = \frac{2}{1 + \sqrt{1 - (\rho(J))^2}}$$ where $J = I - D^{-1}A$ is the "G" matrix for Jacobi.

4.7 Steepest descent and conjugate gradient methods

To solve $A\vec{x} = \vec{b}$ where A is sym. posdef.

<u>Lemma</u> The unique sol'n to $A\vec{x} = \vec{b}$ is the unique minimizer of the quadratic form

$\langle \vec{x}, \vec{y} \rangle := \vec{x}^T \vec{y}$

$$q(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle - 2\langle \vec{x}, \vec{b} \rangle$$

$\nabla q(\vec{x}) = 2A\vec{x} - 2\vec{b}$

<u>Proof</u> $q(\vec{x} + t\vec{v}) = \langle \vec{x} + t\vec{v}, A\vec{x} + tA\vec{v} \rangle - 2\langle \vec{x} + t\vec{v}, \vec{b} \rangle$

$$= \langle \vec{x}, A\vec{x} \rangle - 2\langle \vec{x}, \vec{b} \rangle + 2t\langle \vec{v}, A\vec{x} - \vec{b} \rangle + t^2 \langle \vec{v}, A\vec{v} \rangle$$

$ax^2 + bx + c$ $a > 0$

$x = -\frac{b}{2a}$

(taking $\vec{v} \neq \vec{0}$)

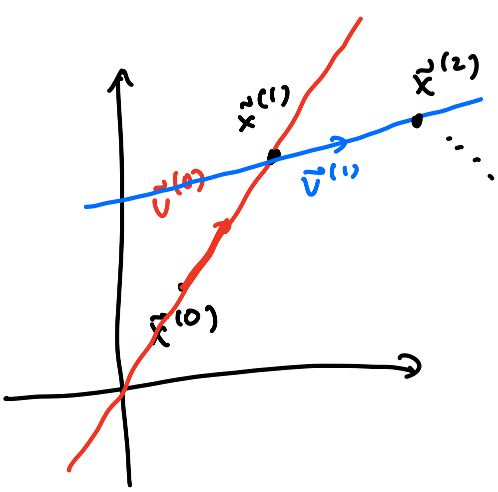
① If $A\vec{x} = \vec{b}$, then $q(\vec{x} + t\vec{v}) > q(\vec{x})$ $\forall t \neq 0$.

$\Rightarrow$ $\vec{x}$ is the unique minimizer of q.

② If $A\vec{x} \neq \vec{b}$, then $\min_{t \in \mathbb{R}} q(\vec{x} + t\vec{v})$ is achieved when

$$t = \frac{\langle \vec{v}, \vec{b} - A\vec{x} \rangle}{\langle \vec{v}, A\vec{v} \rangle}.$$

- This gives a general iterative algorithm to find minimizer of $q(\vec{x})$.



At k-th iteration, choose a search direction $\vec{v}^{(k)} \neq \vec{0}$, then let

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + t_k \vec{v}^{(k)}$$ where

$$t_k = \frac{\langle \vec{v}^{(k)}, \vec{b} - A\vec{x}^{(k)} \rangle}{\langle \vec{v}^{(k)}, A\vec{v}^{(k)} \rangle}.$$

Steepest descent

Choose $\vec{v}^{(k)} = \nabla q(\vec{x}^{(k)}) \parallel \vec{b} - A\vec{x}^{(k)} = \vec{r}^{(k)}$

$2(A\vec{x} - \vec{b})$

Proof of convergence of SOR, continued

$$\frac{1}{1-\lambda} \vec{x}^* A \vec{x} = \vec{x}^* (\omega^{-1} D - C_L) \vec{x}$$

$$\mathrm{Re} \frac{1}{1-\lambda} \vec{x}^* A \vec{x} = \vec{x}^* \left((\omega^{-1} - \tfrac{1}{2}) D + \tfrac{1}{2} A \right) \vec{x}$$

$$\left(\mathrm{Re} \frac{1}{1-\lambda} - \frac{1}{2} \right) \underbrace{\vec{x}^* A \vec{x}}_{>0} = \underbrace{(\omega^{-1} - \tfrac{1}{2})}_{>0} \underbrace{\vec{x}^* D \vec{x}}_{>0}$$

$$\Rightarrow \mathrm{Re} \frac{1}{1-\lambda} - \frac{1}{2} > 0$$

$\lambda = a + bi$ $\qquad \dfrac{1}{1-a-bi} = \dfrac{1-a+bi}{(1-a)^2 + b^2} \quad \Rightarrow \quad \dfrac{1-a}{(1-a)^2 + b^2} > \dfrac{1}{2}$

$$2 - 2a > 1 - 2a + a^2 + b^2$$

$$a^2 + b^2 < 1 \quad \Rightarrow \quad |\lambda| < 1$$