4.4 (continued)

Condition number

Question: when solving $A\vec{x} = \vec{b}$, if $\vec{b}$ is perturbed, how does it affect the soln?

Suppose $\vec{x}$ is the soln to $A\vec{x} = \vec{b}$, and $\tilde{\vec{x}}$ is the soln to $A\tilde{\vec{x}} = \tilde{\vec{b}}$, then

$$A(\vec{x} - \tilde{\vec{x}}) = \vec{b} - \tilde{\vec{b}}$$

$$\vec{x} - \tilde{\vec{x}} = A^{-1}(\vec{b} - \tilde{\vec{b}})$$

$$\|\vec{x} - \tilde{\vec{x}}\| \leq \|A^{-1}\| \cdot \|\vec{b} - \tilde{\vec{b}}\|$$ for some vector/matrix norm.

To measure in relative error,

$$\|\vec{b}\| \leq \|A\| \cdot \|\vec{x}\|$$

$$\frac{1}{\|\vec{x}\|} \leq \|A\| \cdot \frac{1}{\|\vec{b}\|}$$

$$\Rightarrow \frac{\|\vec{x} - \tilde{\vec{x}}\|}{\|\vec{x}\|} \leq \underbrace{\|A\| \cdot \|A^{-1}\|}_{:= k(A)} \cdot \frac{\|\vec{b} - \tilde{\vec{b}}\|}{\|\vec{b}\|}$$

"condition number of A"

$b_0 = 1.12345 \times 10^{\square}$

$x_0 = 2.31478 \times 10^{\square}$

if $k(A) \approx 10^?$

- A matrix A is **ill-conditioned** if $k(A)$ is "large". Otherwise it's **well-conditioned**

Ex $A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}$ $\quad A^{-1} = \varepsilon^{-2} \begin{pmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{pmatrix}$

In ℓ^1-norm, $\|A\| = 2+\varepsilon$ $\quad \|A^{-1}\| = \varepsilon^{-2}(2+\varepsilon)$

$$\kappa(A) = \varepsilon^{-2}(2+\varepsilon)^2$$

Thm Denote $\vec{e} = \vec{x} - \tilde{\vec{x}}$, $\vec{r} = \vec{b} - \tilde{\vec{b}}$

$$\frac{1}{\kappa(A)} \frac{\|\vec{r}\|}{\|\vec{b}\|} \le \frac{\|\vec{e}\|}{\|\vec{x}\|} \le \kappa(A) \frac{\|\vec{r}\|}{\|\vec{b}\|}$$

$A\vec{x} = \vec{b}$

$\vec{x} = A^{-1}\vec{b}$

4.5 Neumann series, iterative refinement

Let $\|\cdot\|$ be fixed vector/matrix norm.

Thm If A is $n \times n$ w/ $\|A\| < 1$, then $I - A$ is invertible and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$ $\quad |x| < 1$

↳ means $\lim_{m\to\infty} \sum_{k=0}^{m} A^k$ in the sense of $\|\cdot\|$

i.e., $\lim_{m\to\infty} \left\| (I-A)^{-1} - \sum_{k=0}^{m} A^k \right\| = 0$.

Proof ① Prove that $\lim_{m\to\infty} \sum_{k=0}^{m} A^k$ exists.

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$(A+B)^2 = A^2 + AB + BA + B^2$

For $m_1 > m_2 > M$,

$$\left\| \sum_{k=0}^{m_1} A^k - \sum_{k=0}^{m_2} A^k \right\| = \left\| \sum_{k=m_2+1}^{m_1} A^k \right\| \le \sum_{k=m_2+1}^{m_1} \|A^k\|$$

$$\le \sum_{k=m_2+1}^{m_1} \|A\|^k \le \sum_{k=M}^{\infty} \|A\|^k = \|A\|^M \cdot \frac{1}{1-\|A\|}$$

is arbitrarily small for large M

$\Rightarrow$ partial sums are a Cauchy sequence.

(3) Prove that $(I-A)\sum_{k=0}^{\infty} A^k = I$.

$$(I-A)\sum_{k=0}^{\infty} A^k = (I-A)\lim_{m\to\infty}\sum_{k=0}^{m} A^k = \lim_{m\to\infty}(I-A)\sum_{k=0}^{m} A^k$$

$$= \lim_{m\to\infty}\left(I - A^{m+1}\right) = I. \qquad \|A^{m+1}\| \le \|A\|^{m+1} \to 0$$

- It follows that $\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$

If you solve $(I-A)\vec{x} = \vec{b}$ w/ $\|A\|$ small, then it's well-conditioned.

Iterative refinement

Suppose we solve $A\vec{x} = \vec{b}$ by a numerical method and get an approximate sol'n $\vec{x}^{(0)}$

Then, denote $\vec{e}^{(0)} = \vec{x} - \vec{x}^{(0)}$

$$\Rightarrow \quad A\vec{e}^{(0)} = A\vec{x} - A\vec{x}^{(0)} = \vec{b} - A\vec{x}^{(0)} = \vec{r}^{(0)}$$

Then we solve for $\vec{e}^{(0)}$ from $A\vec{e}^{(0)} = \vec{r}^{(0)}$ numerically, get $\tilde{\vec{e}}^{(0)}$

define $\vec{x}^{(1)} = \vec{x}^{(0)} + \tilde{\vec{e}}^{(0)}$

This can be applied iteratively.

- To make it work, one needs to use higher precision for calculating $\vec{r}^{(0)} = \vec{b} - A\vec{x}^{(0)}$ because it's a difference of two close numbers.

Iterative refinement for Gaussian elimination is cheap ($O(n^2)$ each step if LU-decomp. for A is saved).

- Error analysis: Suppose the numerical solver for $A\vec{x} = \vec{b}$ produces $\vec{x} = B\vec{b}$ where B is an approximation of A^{-1}.

Then $\vec{x}^{(0)} = B\vec{b}$

$$\vec{x}^{(1)} = \vec{x}^{(0)} + B(\vec{b} - A\vec{x}^{(0)}) = B\vec{b} + B(\vec{b} - AB\vec{b})$$

$$= B(\vec{b} + \vec{b} - AB\vec{b}) = B(I + (I - AB))\vec{b}$$

$$\vec{x}^{(2)} = \vec{x}^{(1)} + B(\vec{b} - A\vec{x}^{(1)})$$

$$= B(I + (I - AB))\vec{b} + B(\vec{b} - AB(I + (I - AB))\vec{b})$$

$$= B\left((I + (I - AB))\vec{b} + \vec{b} - AB(I + (I - AB))\vec{b}\right)$$

$$= B(I + (I - AB) + I - AB - AB(I - AB))\vec{b}$$

$$= B(I + (I - AB) + (I - AB)^2)\vec{b}$$

...

$$\vec{x}^{(m)} = B\sum_{k=0}^{m}(I - AB)^k\vec{b} \qquad \text{(can be proved by induction)}$$

If $\|I - AB\| < 1$ then $\sum_{k=0}^{m}(I - AB)^k \to [I - (I - AB)]^{-1} = (AB)^{-1}$

$$= B^{-1}A^{-1}$$

$$\Rightarrow \quad \vec{x}^{(m)} \to A^{-1}\vec{b}$$