4.4 Norms, error analysis

Suppose we solve $A\vec{x} = \vec{b}$ numerically. If $\vec{x}$ is the exact sol'n and $\tilde{\vec{x}}$ is a numerical approximation, to measure the error:

$$\begin{cases} \text{size of } \vec{x} - \tilde{\vec{x}} & \text{"error vector"} \\ \text{size of } A\tilde{\vec{x}} - \vec{b} & \text{"residue"} \end{cases}$$

Def Let V be a vector space. A norm on V is a function

$\|\cdot\| : V \to [0, \infty)$ s.t.

1. $\|x\| \geq 0 \quad \forall x \in V$; "$=$" only when $x = 0$
2. $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall x \in V, \lambda \in \mathbb{R}$
3. "triangle inequality" $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$.

Ex The Euclidean norm on $\mathbb{R}^n$:

$$\|\vec{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

More generally, the ℓ^p-norm of a vector $\vec{x} \in \mathbb{R}^n$ is

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad 1 \leq p < \infty$$

ℓ^∞-norm: $\quad \|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

- Fact: all norms on $\mathbb{R}^n$ (finite dim. vector space) are equivalent, i.e., if $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ are two norms on $\mathbb{R}^n$, then $\exists\ c, C > 0$ s.t.

$$c\|\vec{x}\|_{(1)} \le \|\vec{x}\|_{(2)} \le C\|\vec{x}\|_{(1)} \qquad \forall\ \vec{x} \in \mathbb{R}^n$$

(only true for finite-dim vector spaces)

However, when n is large, the constants c, C may get extreme.

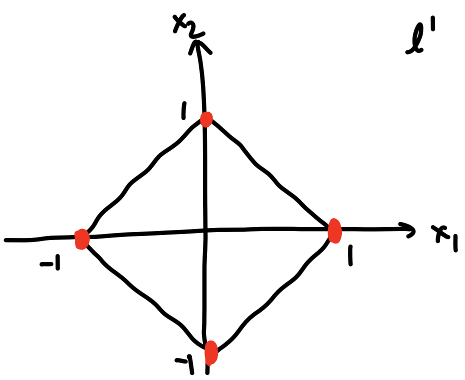
For example, consider $\|\cdot\|_1$, $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

$$\|\vec{x}\|_\infty \le \|\vec{x}\|_1 \le n\|\vec{x}\|_\infty$$

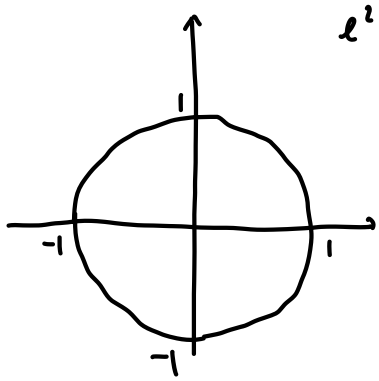
$\hookrightarrow$ gets larger when dim is large.

	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
$(1,0,\cdots,0)$	1	1	1
$(1,1,\cdots,1)$	n	$\sqrt{n}$	1

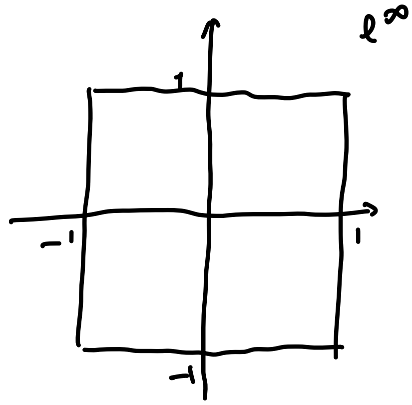
- Unit balls in $\mathbb{R}^2$ $\qquad \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \le 1\}$



$|x_1| + |x_2| \le 1$



$x_1^2 + x_2^2 \le 1$



$\max\{|x_1|, |x_2|\} \le 1$

<u>Def</u> Given a norm $\|\cdot\|$ on $\mathbb{R}^n$. The <u>matrix norm subordinate to</u> $\|\cdot\|$ is defined by

$$\|A\| = \sup\{\|A\vec{u}\| : \vec{u} \in \mathbb{R}^n, \|\vec{u}\| = 1\}.$$

"operator norm"

where A is an $n\times n$ matrix

- Basic properties

(1) $\|A\vec{x}\| \leq \|A\| \cdot \|\vec{x}\|$ (direct consequence of def)

(2) $\|\cdot\|$ (matrix norm) is a norm on the vector space of $n\times n$ matrices.

(3) $\|AB\| \leq \|A\| \cdot \|B\|$

Proof of (2) ③

Need to prove : $\|A+B\| \leq \|A\| + \|B\|$

For any $\vec{u} \in \mathbb{R}^n$ w/ $\|\vec{u}\| = 1$,

$$\|(A+B)\vec{u}\| = \|A\vec{u} + B\vec{u}\| \leq \|A\vec{u}\| + \|B\vec{u}\| \leq \|A\| + \|B\|$$

$$\Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

Thm Let $A = (a_{ij})$ be an $n\times n$ matrix. Its matrix norms subordinate to ℓ^1, ℓ^2, ℓ^∞ are

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}| = C_1 \quad \text{(largest column sum)}$$

$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}| = C_\infty \quad \text{(largest row sum)}$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$$

Proof For $\|\cdot\|_1$:

① For any $\vec{u} \in \mathbb{R}^n$ w/ $\|\vec{u}\|_1 = 1$, (need to prove $\|A\vec{u}\|_1 \leq C_1$)

$$\|A\vec{u}\|_1 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} u_j \right| \leq \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \cdot |u_j|$$

$$= \sum_{j=1}^n \underbrace{\sum_{i=1}^n |a_{ij}|} \cdot |u_j| \leq \sum_{j=1}^n C_1 |u_j| = C_1$$

② Need to find $\vec{u} \in \mathbb{R}^n$ w/ $\|\vec{u}\|_1 = 1$ s.t. $\|A\vec{u}\|_1 = C_1$

$$\underbrace{\begin{pmatrix} & \overset{j}{|} & \\ & | & \\ & | & \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{u}} \begin{array}{l} \\ \\ \\ j \\ \\ \\ \\ \end{array} = \begin{pmatrix} | \\ | \\ | \end{pmatrix}$$

In expression of C_1, let j be the index which achieves the max.

Take $\vec{u} = \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ j \\ \\ \\ \\ \end{array}$

Then $A\vec{u} =$ j-th column of A

$$\|A\vec{u}\| = \sum_{i=1}^{n} |a_{ij}| = C_1$$

For $\|\cdot\|_\infty$

① For any $\vec{u} \in \mathbb{R}^n$ w/ $\|\vec{u}\|_\infty = 1$,

$$\|A\vec{u}\|_\infty = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} u_j \right| \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \cdot |u_j|$$

$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = C_\infty$$

② Need to find $\vec{u} \in \mathbb{R}^n$ w/ $\|\vec{u}\|_\infty = 1$ s.t. $\|A\vec{u}\|_\infty = C_\infty$

$$i \underbrace{\begin{pmatrix} \text{———} \\ \\ \\ \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{\vec{u}} = \begin{pmatrix} \bullet \\ \\ \\ \end{pmatrix}$$

In expression of C_∞, let i be the index which achieves the max.

Take $\vec{u}$ as $u_j = \mathrm{sign}(a_{ij})$

Then $(A\vec{u})_i = \sum_{j=1}^{n} a_{ij} u_j = \sum_{j=1}^{n} |a_{ij}| = C_\infty$

$$\begin{pmatrix} 2 & 3 & -4 \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \boxed{1} \\ \boxed{1} \\ \boxed{-1} \end{pmatrix} = \begin{pmatrix} \bullet \\ \\ \end{pmatrix}$$

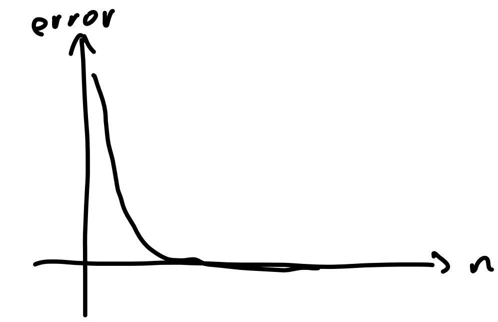
For $\|\cdot\|_2$,

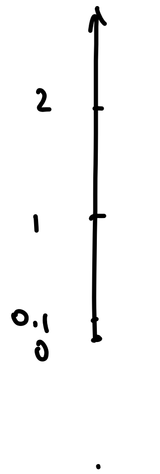
$$\|A\vec{u}\|_2^2 = (A\vec{u})^T(A\vec{u}) = \vec{u}^T \underbrace{A^T A}\, \vec{u}$$

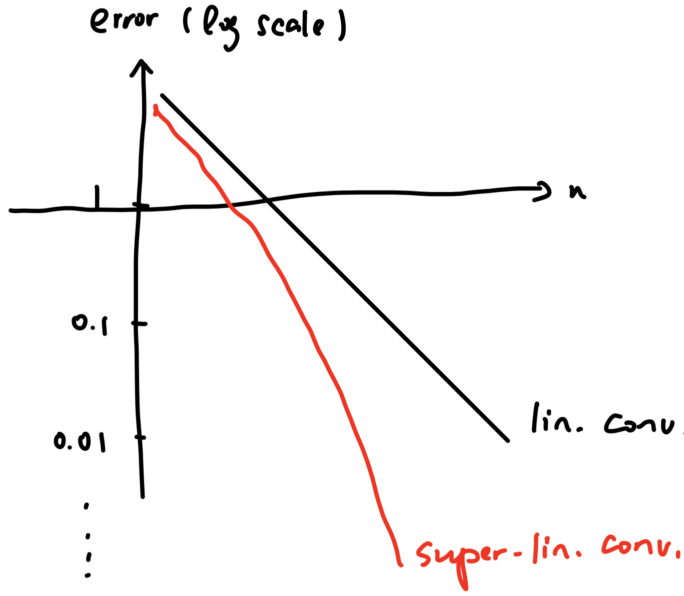
Generally, if B is $n \times n$ sym. semi-pos def, then

$$\max_{\|\vec{u}\|_2 = 1} \vec{u}^T B \vec{u} = \lambda_{max}(B)$$

$$B = P^T D P \qquad P \text{ orthogonal}$$







semilog y

semilog x

loglog

$$\text{error} = C \lambda^n$$

$$\log(\text{error}) = \log C + n \underbrace{\log \lambda}_{\text{slope}}$$