Gaussian elimination w/ scaled row pivoting

**Ex**

$$\left(\begin{matrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ \boxed{3} & -2 & 1 \end{matrix}\right) \begin{matrix} \text{ratio} \\ 2/6 \\ 1/8 \\ \underline{3/3} \end{matrix} \qquad p = (1\ 2\ 3) \qquad s = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix}$$

$$\left(\begin{matrix} \frac{2}{3} & \boxed{\frac{13}{3}} & -\frac{20}{3} \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ 3 & -2 & 1 \end{matrix}\right) \begin{matrix} \frac{13}{3}/6 \\ \frac{16}{3}/8 \\ \phantom{0} \end{matrix} \qquad p = (3\ 2\ 1)$$

$$\begin{pmatrix} \frac{2}{3} & \frac{13}{3} & -\frac{20}{3} \\ \frac{1}{3} & -\frac{16}{13} & -\frac{7}{13} \\ 3 & -2 & 1 \end{pmatrix} \qquad p = (3\ 1\ 2)$$

$$P_{ij} = \delta_{p_i j} \qquad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad PA = LU.$$

$$U = \begin{pmatrix} 3 & -2 & 1 \\ & \frac{13}{3} & -\frac{20}{3} \\ & & -\frac{7}{13} \end{pmatrix} \qquad L = \begin{pmatrix} 1 & & \\ \frac{2}{3} & 1 & \\ \frac{1}{3} & -\frac{16}{13} & 1 \end{pmatrix}$$

- Extra cost from scaled row pivoting

  initial : $O(n^2)$ comparisons

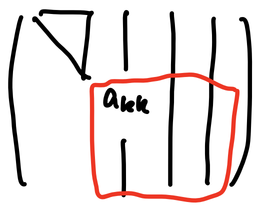
  every step : $O(n)$ divisions, comparisons $\longrightarrow$ total $O(n^2)$ divisions, comparisons.

  $n + (n-1) + \cdots + 2$

$\Rightarrow$ total cost $O(n^2)$ negligible compared to G. e. cost $O(n^3)$

- Other pivoting strategies:
  - partial pivoting: $k$-th step, choose $i$-th row w/ $|a_{ik}|$ largest

    simpler than scaled row pivoting but only applicable if $\{a_{ij}\}$ are about the same size.
  - complete pivoting: $k$-th step, choose the largest $|a_{ij}|$ and take it as pivot by switching rows and columns.

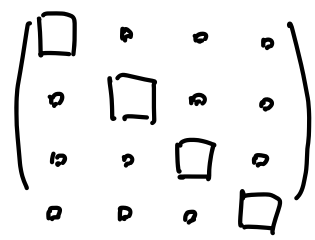



It makes round-off effects smallest, but more expensive. (cost $O(n^3)$)

## Diagonally dominant matrix

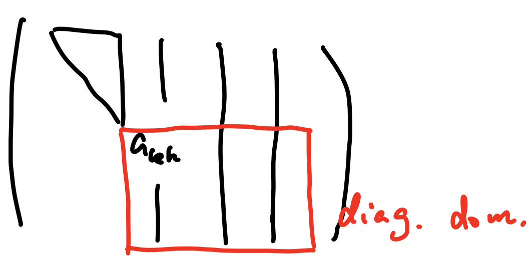
Def An $n\times n$ matrix $A=(a_{ij})$ is (strict) diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad 1\le i\le n$$



Thm Gaussian elimination w/o pivoting preserves diagonal dominance.

- Gaussian elimination can always be done for diagonally dominant matrix $A$. Therefore, $A$ is nonsingular and has LU-decomposition.
- Scaled row pivoting would never do exchanges if $\{s_i\}$ were updated every step.



$\Rightarrow$ you don't need to turn on pivoting for diagonally dominant matrix.

Proof of thm It suffice to do Step 1

$$\begin{pmatrix} \boxed{a_{11}} & a_{12} & \cdot\, a_{1j} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & a_{ij} & \vdots \\ a_{i1} & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & & \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}$$

need to prove $|a_{ii}^{(2)}| > \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{ij}^{(2)}| \quad i=2,\dots,n.$

$$a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \qquad a_{ii}^{(2)} = a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i}$$

$$\sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{ij}^{(2)}| = \sum_{\substack{j=2 \\ j\neq i}}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \le \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{ij}| + \frac{|a_{i1}|}{|a_{11}|} \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{1j}|$$

know: $|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = |a_{i1}| + \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{ij}|$

$\Rightarrow \quad \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{ij}| < |a_{ii}| - |a_{i1}|$

$$|a_{11}| > \sum_{j=2}^{n} |a_{1j}| = \sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{1j}| + |a_{1i}|$$

Similarly, $\sum_{\substack{j=2 \\ j\neq i}}^{n} |a_{1j}| < |a_{11}| - |a_{1i}|$

$$< |a_{ii}| - \cancel{|a_{i1}|} + \frac{|a_{i1}|}{|a_{11}|}\left(\cancel{|a_{11}|} - |a_{1i}|\right)$$

$$= |a_{ii}| - \frac{|a_{i1}|}{|a_{11}|} |a_{1i}| \le \left| a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i} \right| = |a_{ii}^{(2)}|$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1} \end{pmatrix} \qquad \begin{pmatrix} \boxed{1} & 0 & \boxed{1} \\ 0 & 1 & 0 \\ \boxed{1} & 0 & \boxed{1} \end{pmatrix}$$

- All results are true for <u>irreducibly diagonally dominant</u> matrices,

i.e. $\begin{cases} |a_{ii}| \geq \hat{\sum\limits_{\substack{j=1 \\ j\neq i}}} |a_{ij}|, \ i=1,\cdots,n & \text{w/ at least one "}>\text{"} \\ \text{"irreducible": there is no nonempty proper subset } S \subset \{1,\cdots,n\} \\ \qquad \text{s.t. } a_{ij}=0 \text{ for any } i\in S, j\notin S. \end{cases}$

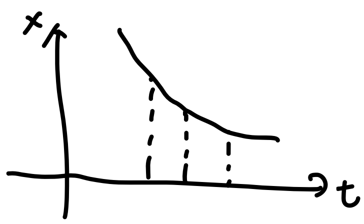
$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

$i \leftrightarrow j$ if $a_{ij} \neq 0$

$1 \leftrightarrow 2 \leftrightarrow 3 \cdots\cdots n-1 \leftrightarrow n$

$A\vec{x} = \vec{b}$ $\qquad -x_1 + 2x_2 - x_3 = b_2 \quad \cdots$

$x(t)$



## G. e. for tri-diagonal matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & a_{23} & & & \\ & a_{32} & a_{33} & a_{34} & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{nn} \end{pmatrix}$$

$a_{ij}=0 \ \forall |i-j|>$

$\Big\downarrow$ G.e. w/o pivoting, Step 1

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & & & \\ 0 & \boxed{\begin{matrix} a_{22}^{(2)} & a_{23} & & \\ a_{32} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \end{matrix}} \end{pmatrix}$$

tri-diag.

Only need to compute

$$a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$\downarrow \vdots$$

$$L = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ & \ddots & \ddots & \\ & & l_{n,n-1} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} u_{11} & u_{12} & & & \\ & u_{22} & u_{23} & & \\ & & \ddots & \ddots & \\ & & & u_{n-1,n-1} & u_{n-1,n} \\ & & & & u_n \end{pmatrix}$$

$L\vec{y} = \vec{b}$

$y_1 = b_1$

$y_2 = b_2 - l_{21} y_1$

$y_3 = b_3 - l_{32} y_2$

$\vdots$

$U\vec{x} = \vec{y}$

$x_n = \frac{1}{u_n} y_n$

$x_{n-1} = \frac{1}{u_{n-1,n-1}} \left( y_{n-1} - u_{n-1,n} x_n \right)$

$x_{n-2} = \frac{1}{u_{n-2,n-2}} \left( y_{n-2} - u_{n-2,n-1} x_{n-1} \right)$

$\vdots$

Total cost: (multip. div.)

$$\begin{cases} \text{G. e.} & 2(n-1) \\ L\vec{y} = \vec{b} & (n-1) \\ U\vec{x} = \vec{y} & (2n-1) \end{cases}$$