4.2 (continued)

Recall:

- If A is $n \times n$ symmetric, then, if A has an LU-decomposition, then $A = LDL^T$ where L has diagonal entries 1
- If all leading principles minors of A are nonsingular, then A has an LU-decomposition.

<u>Thm</u> If A is symmetric and positive-definite, then A can be written as $A = LL^T$ where L is lower-triangular w/ diagonal entries positive. (called <u>Cholesky factorization</u>)

<u>Proof</u>

Recall: an $n \times n$ sym A is positive-definite if $\vec{v}^T A \vec{v} > 0 \quad \forall \vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$.

- Any leading principle minor of A is sym. pos. def.; in particular, it's nonsingular.

$$\begin{array}{cc} & k \\ k & \begin{pmatrix} \boxed{A_k} & \\ & \end{pmatrix} \end{array} \qquad \vec{v} \in \mathbb{R}^k, \ \vec{v} \neq \vec{0} \qquad \tilde{\vec{v}} = \begin{pmatrix} \vec{v} \\ \vec{0} \end{pmatrix}_{n \times 1} \in \mathbb{R}^n$$

$$\vec{v}^T A_k \vec{v} = \tilde{\vec{v}}^T A \tilde{\vec{v}} > 0$$

$\Rightarrow$ A has an LU-decomposition

$\Rightarrow$ $A = L \underbrace{D}_{\text{pos. def.}} L^T$ w/ L diagonal entries 1

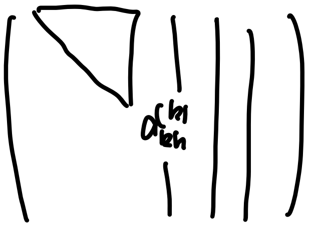
$D = \operatorname{diag}\{d_1, \dots, d_n\}$ $\quad d_i > 0$

$$D^{1/2} = \operatorname{diag}\{\sqrt{d_1}, \dots, \sqrt{d_n}\}$$

$$A = LD^{1/2}(D^{1/2})^T L^T = (LD^{1/2})(LD^{1/2})^T$$

4.3 pivoting

In k-th step of Gaussian elimination, if $a_{kk}^{(k)} = 0$, then one needs to do row exchange.



$$\left(\begin{array}{cc:c} 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{\text{①} \leftrightarrow \text{②}} \left(\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

$x_2 = 1 \qquad x_1 = 2 - 1 = 1$

- If $|a_{kk}^{(k)}|$ is small, then it's also desired to do row exchange due to round-off error.

Ex Consider 3 effective decimal digits.

$$\left(\begin{array}{cc:c} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{\text{②} \to \text{②} - 10000 \times \text{①}} \left(\begin{array}{cc:c} 0.0001 & 1 & 1 \\ 0 & \underset{-10000}{\cancel{-9999}} & \underset{-10000}{\cancel{-9998}} \end{array}\right)$$

$x_2 = \dfrac{-10000}{-10000} = 1 \qquad x_1 = \dfrac{1}{0.0001}(1-1) = 0$

far from exact sol'n

$$\xrightarrow{\text{①} \leftrightarrow \text{②}} \left(\begin{array}{cc:c} 1 & 1 & 2 \\ 0.0001 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\text{②} \to \text{②} - 0.0001 \times \text{①}} \left(\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & \underset{1}{\cancel{0.9999}} & \underset{1}{\cancel{0.9998}} \end{array}\right)$$

$x_2 = 1 \qquad x_1 = 2 - 1 = 1$

Exact sol'n $\begin{cases} x_1 = 1.0001\cdots \\ x_2 = 0.9998\cdots \end{cases}$

- General principle about pivoting: try to make $|a_{kk}^{(k)}|$ "large."

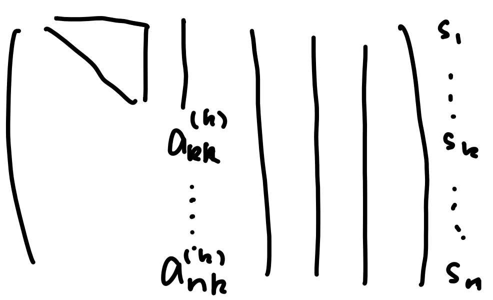
$$\left(\begin{array}{cc|c} 1 & 10000 & 10000 \\ 1 & 1 & 2 \end{array}\right)$$

Scaled row pivoting

- Idea: it's problematic if $\left| \dfrac{a_{kk}^{(k)}}{a_{kj}^{(k)}} \right|$ is small for some $j \in \{k+1, \cdots, n\}$

- Define $s_i = \max\limits_{1 \le j \le n} |a_{ij}|$ "scale of i-th row"

  In k-th step, choose the pivoting row i so that $\dfrac{|a_{ik}^{(k)}|}{s_i}$ is the largest among $k \le i \le n$



- LU factorization w/ row pivoting.

Recall: w/o pivoting:

$A^{(2)} = M^{(1)} A^{(1)}, \quad A^{(3)} = M^{(2)} A^{(2)} \quad \cdots \quad A^{(n)} = M^{(n-1)} A^{(n-1)}$

$A^{(n)}$: U


w/ pivoting:

$$A^{(2)} = M^{(1)} P^{(1)} A^{(1)}, \quad A^{(3)} = M^{(2)} P^{(2)} A^{(2)}, \quad \dots$$

$$P^{(1)} = \begin{pmatrix} 0 & & & & 1 & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ 1 & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \qquad P^{(2)} = \begin{pmatrix} 1 & & & & & & & & \\ & 0 & & & & 1 & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & 1 & & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

i_1-th $\qquad$ i_2-th

$$A^{(3)} = M^{(2)} P^{(2)} M^{(1)} P^{(1)} A^{(1)} = M^{(2)} \tilde{M}^{(1)} P^{(2)} P^{(1)} A^{(1)}$$

$$P^{(2)} \underbrace{\begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ * & & \ddots & & \\ * & & & \ddots & \\ * & & & & 1 \end{pmatrix}}_{M^{(1)}} = \begin{pmatrix} 1 & & & & & & \\ * & 0 & & & 1 & & \\ * & & 1 & & & & \\ * & & & \ddots & & & \\ * & & & & 1 & & \\ * & 1 & & & 0 & & \\ * & & & & & 1 & \\ * & & & & & & \ddots \\ * & & & & & & & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & & \ddots & \\ * & & & 1 \end{pmatrix}}_{\tilde{M}^{(1)}} P^{(2)} = \begin{pmatrix} 1 & & & & & & \\ * & 0 & & & 1 & & \\ * & & 1 & & & & \\ * & & & \ddots & & & \\ * & & & & 1 & & \\ * & 1 & & & 0 & & \\ * & & & & & 1 & \\ * & & & & & & \ddots \\ * & & & & & & & 1 \end{pmatrix}$$

$$P^{(2)} M^{(1)} = \tilde{M}^{(1)} P^{(2)}$$

Finally $$U = A^{(n)} = \underbrace{M^{(n-1)} M^{(n-2)} \cdots M^{(1)}}_{\substack{\text{modified} \\ := L^{-1}}} \underbrace{P^{(n-1)} \cdots P^{(1)}}_{P} A$$

$$\Rightarrow \quad PA = LU$$

Thm For any nonsingular $n \times n$ matrix A, the above algorithm gives the factorization

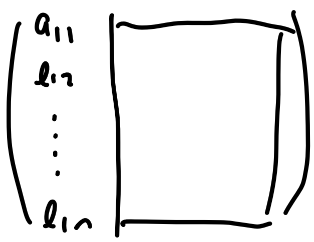
$$PA = LU$$

permutation matrix $\qquad$ lower-tri. $\qquad$ upper-tri.

To solve $A\vec{x} = \vec{b}$:

$$
\begin{aligned}
PA\vec{x} &= P\vec{b} \\
LU\vec{x} &= P\vec{b} \\
&\begin{cases} L\vec{y} = P\vec{b} \\ U\vec{x} = \vec{y} \end{cases}
\end{aligned}
$$

- Realization in computer
  - keep track of P by an array starting from $p = (1\ 2\ 3\ \cdots\ n)$
  - save l_{ij} values in the corresponding zero positions in A



  - don't actually need to switch rows of A in the storage.

Ex
$$
\begin{pmatrix} 2 & 3 & -6 \\ 1 & -6 & 8 \\ \boxed{3} & -2 & 1 \end{pmatrix}
\begin{array}{c} \text{ratio} \\ 2/6 \\ 1/8 \\ 3/3 \end{array}
\qquad p = (1\ 2\ 3) \qquad s = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix}
$$

$$
\begin{pmatrix} \frac{2}{3} & \frac{13}{3} & -\frac{20}{3} \\ \frac{1}{3} & -\frac{16}{3} & \frac{23}{3} \\ 3 & -2 & 1 \end{pmatrix}
\begin{array}{c} \frac{13}{3}/6 \\ \frac{16}{3}/8 \end{array}
\qquad p = (3\ 2\ 1)
$$