3.5 (continued)

Bairstow's method

(find possibly complex roots of a real polynomial)

Let $p(z)$ be a real polynomial. If $w = x + iy$ is a complex zero w/ $y \neq 0$, then

$$p(w) = 0 \rightsquigarrow p(\bar{w}) = 0$$

i.e. $\bar{w} = x - iy$ is also a zero of $p(z)$.

$$a_n w^n + a_{n-1} w^{n-1} + \cdots + a_1 w + a_0$$

Therefore $p(z)$ has a factor $(z - w)(z - \bar{w}) = z^2 - (w + \bar{w}) z + w\bar{w}$

real coefficients.

$\rightsquigarrow$ we only need to find factors of the form

$$z^2 + uz + v, \quad u, v \in \mathbb{R}.$$

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = (z^2 + uz + v)(b_n z^{n-2} + \cdots + b_2) + b_1(z + u) + b_0$$

deg n : $b_n = a_n$

deg $n-1$: $b_{n-1} = a_{n-1} - u b_n$

deg $n-2$: $b_{n-2} = a_{n-2} - u b_{n-1} - v b_n$

$\vdots$

deg 1 : $b_1 = a_1 - u b_2 - v b_3$

deg 0 : $b_0 = a_0 - u b_1 - v b_2$

iterative calculation

$\Rightarrow$ need $\begin{cases} b_0(u,v) = 0 \\ b_1(u,v) = 0 \end{cases}$ Apply multi-D Newton's method.

To calculate $\frac{\partial b_0}{\partial u}, \dots$, need:

$$b_n = a_n \qquad\qquad \frac{\partial b_n}{\partial u} = 0$$

$$b_{n-1} = a_{n-1} - u b_n \qquad \xrightarrow{\frac{\partial}{\partial u}} \qquad \frac{\partial b_{n-1}}{\partial u} = -b_n - u\frac{\partial b_n}{\partial u}$$

$$b_{n-2} = a_{n-2} - u b_{n-1} - v b_n \qquad\qquad \frac{\partial b_{n-2}}{\partial u} = -b_{n-1} - u\frac{\partial b_{n-1}}{\partial u} - v\frac{\partial b_n}{\partial u}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$b_1 = a_1 - u b_2 - v b_3$$

$$b_0 = a_0 - u b_1 - v b_2$$

- Bairstow's method has quadratic convergence for simple zeros

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Laguerre's method

$$A = -\frac{p'(z)}{p(z)} \qquad\qquad n = \deg(p)$$

$$B = A^2 - \frac{p''(z)}{p(z)}$$

$$C = \frac{1}{n}\left(A \pm \sqrt{(n-1)(nB - A^2)}\right)$$

$$z_{new} = z + \frac{1}{C}$$

choose $\pm$ to make $|C|$ largest.

- Intuitive derivation

$$p(z) = a(z-r_1)\cdots(z-r_n)$$

$$A = -\frac{a(z-r_2)\cdots(z-r_n) + a(z-r_1)(z-r_3)\cdots(z-r_n) + \cdots}{a(z-r_1)\cdots(z-r_n)}$$

$$= \frac{1}{r_1 - z} + \cdots + \frac{1}{r_n - z}$$

$$\frac{dA}{dz} = -\frac{p''(z)p(z) - p'(z)^2}{p(z)^2} = B = \frac{1}{(r_1 - z)^2} + \cdots + \frac{1}{(r_n - z)^2}$$

If z is close to one of the zeros, say, r_1, and far from others, then we make approximation

$$v \approx \frac{1}{r_k - z} \qquad k = 2, \cdots, n$$

$$u = \frac{1}{r_1 - z}$$

$$A = u + (n-1)v$$
$$B = u^2 + (n-1)v^2$$
$\rightsquigarrow$ Solve for u, v, we get $u = C$

- Error analysis (heuristic)

(to prove that it's cubic convergent ...)

Let r be a simple zero assume $|r - z|$ small.

$$A = -\frac{p'(z)}{p(z)}$$

$$B = A^2 - \frac{p''(z)}{p(z)} = \frac{p'(z)^2 - p''(z)p(z)}{p(z)^2}$$

$$C = \frac{1}{n}\left(A \pm \sqrt{(n-1)(nB - A^2)}\right)$$

$$z_{new} = z + \frac{1}{C}$$

$$C = \frac{1}{n}\left(-\frac{p'(z)}{p(z)} \pm \sqrt{(n-1)\frac{(n-1)p'(z)^2 - n\,p''(z)p(z)}{p(z)^2}}\right)$$

$$= -\frac{1}{n} \cdot \frac{p'(z)}{p(z)}\left(1 + \sqrt{(n-1)^2 - (n-1)n\frac{p''(z)p(z)}{p'(z)^2}}\right)$$

$$= -\frac{p'(z)}{p(z)}\left(\frac{1}{n} + \frac{n-1}{n}\sqrt{1 - \frac{n}{n-1}\frac{p''(z)p(z)}{p'(z)^2}}\right)$$

$$\varepsilon = z - r$$

$p(z) = p'\varepsilon + \frac{1}{2}p''\varepsilon^2 + \cdots$ $\quad$ p', p'' refer to evaluated @ r.

$p'(z) = p' + p''\varepsilon + \frac{1}{2}p'''\varepsilon^2 + \cdots$

$p''(z) = p'' + p'''\varepsilon + \frac{1}{2}p''''\varepsilon^2 + \cdots$

$$\frac{1}{c} = -\frac{p(z)}{p'(z)}\left(\frac{1}{n} + \frac{n-1}{n}\sqrt{1 - \frac{n}{n-1}\frac{p''(z)p(z)}{p'(z)^2}}\right)^{-1}$$

discard $O(\varepsilon^3)$ terms.

$$= -\varepsilon\frac{p' + \frac{1}{2}p''\varepsilon}{p' + p''\varepsilon}\left(\frac{1}{n} + \frac{n-1}{n}\sqrt{1 - \frac{n}{n-1}\cdot\frac{p''\,\cancel{p'}\varepsilon}{p'^{\cancel{2}}}}\right)^{-1}$$

$$= -\varepsilon\,\frac{1 + \frac{1}{2}\frac{p''}{p'}\varepsilon}{1 + \frac{p''}{p'}\varepsilon}\left(\frac{1}{n} + \frac{n-1}{n}\left(1 - \frac{1}{2}\cdot\frac{n}{n-1}\cdot\frac{p''}{p'}\varepsilon\right)\right)^{-1}$$

$$= -\varepsilon\underbrace{\left(1 + \frac{1}{2}\frac{p''}{p'}\varepsilon\right)\left(1 - \frac{p''}{p'}\varepsilon\right)\left(1 + \frac{1}{2}\frac{p''}{p'}\varepsilon\right)}$$

$(1+x)^\alpha \approx 1 + \alpha x$

x small

$$= -\varepsilon\left(1 + \frac{1}{2}\frac{p''}{p'}\varepsilon - \frac{p''}{p'}\varepsilon + \frac{1}{2}\frac{p''}{p'}\varepsilon\right) = -\varepsilon$$

- Properties of Laguerre
  - It may converge to complex zeros of real polynomials even if starting from real initial values.
  - If p is real polynomial w/ all roots real, then Laguerre converges monotonically.
  - Empirical evidence: failure of Laguerre's is very rare.
  - Use $|p(z)| < TOL$ for error checking.

- Finding polynomial roots can be <u>ill-conditioned</u>, i.e., a small change in inputs can result in a very large change in outputs.

<u>Ex</u> Wilkinson's polynomial $w(x) = (x-1)(x-2)\cdots(x-20) = x^{20} - \underbrace{210}_{210.000\cdots01} x^{19} + \cdots$

changing x^{19} coefficient by 2^{-23} makes some zeros complex

One can express $p(z)$ in a different way

$$p(z) = \sum_{k=0}^{n} c_k \, b_k(z) \qquad \text{where } \{b_k\} \text{ are some basis polynomials}$$