Problem 1: Rigorously prove the following result: Let $f \in C^3(\mathbb{R})$ and r be a double root of f, i.e., $f(r) = f'(r) = 0$ and $f''(r) \neq 0$. Then, there exists $\delta > 0$ such that for any initial value $x_0 \in (r - \delta, r + \delta)$, Newton's method converges at least linearly to r.

Problem 2: Give a heuristic analysis for order of convergence of the following numerical method for finding a simple root r of $f(x)$: we modify the secant method by utilizing x_{n-2} instead of x_{n-1}, i.e.,

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-2})}{f(x_n) - f(x_{n-2})}$$

Determine whether it is better or worse than the secant method.

Problem 3: Let $f \in C^1(\mathbb{R})$ with $f'(x) > 0$, $\forall x \in \mathbb{R}$, and r is a root of f. For $\alpha > 0$, consider

$$F(x) = x - \alpha f(x)$$

It is clear that r is a fixed point of F.
(1) Rigorously prove the following result: For any $M > 0$, there exists $\alpha_0 > 0$ such that if $\alpha < \alpha_0$ then the functional iteration $x_{n+1} = F(x_n)$ converges to r as long as $x_0 \in [r-M, r+M]$.

(2) Explain why such a functional iteration with small α is not desirable in practice even if it has good convergence property.

Problem 4: An $n \times n$ matrix A is *column diagonally dominant* if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ji}|, \quad i = 1, \ldots, n$$

Prove that column diagonal dominance is preserved by Gaussian elimination if no row exchanges are applied.

Problem 5: An $n \times n$ matrix A is *upper Hessenberg* if

$$a_{ij} = 0, \quad \forall i > j + 1$$

Count the number of multiplication / division in the Gaussian elimination for an upper Hessenberg matrix (assuming no pivoting algorithm is applied).