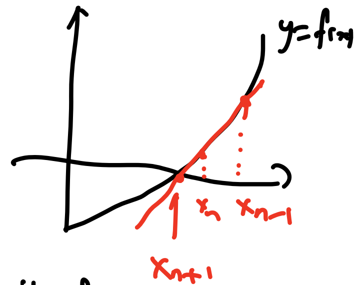
3.3 Secant method

Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (f once, f' once)



Steffensen's method

$$x_{n+1} = x_n - \frac{f(x_n)\, f(x_n)}{f(x_n + f(x_n)) - f(x_n)}$$

(f twice)

Secant method

$$x_{n+1} = x_n - \frac{f(x_n)\,(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad (n \geq 1)$$

(need two initial values x_0, x_1)

(f once)

Error analysis (heuristic) for simple root r.

$e_n := x_n - r$

$$e_{n+1} = e_n - \frac{f(x_n)\,(e_n - e_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$f(x_n) \approx \cancel{f(r)} + f'(r)\, e_n + \frac{1}{2} f''(r)\, e_n^2$$

$$f(x_{n-1}) \approx \cancel{f(r)} + f'(r)\, e_{n-1} + \frac{1}{2} f''(r)\, e_{n-1}^2$$

$$\approx e_n - \frac{\left(f'(r) e_n + \frac{1}{2} f''(r) e_n^2\right)(e_n - e_{n-1})}{f'(r)(e_n - e_{n-1}) + \frac{1}{2} f''(r)\,(e_n^2 - e_{n-1}^2)}$$

$$= \frac{e_n \cdot \frac{1}{2} f''(r)\,(e_n^2 - e_{n-1}^2) - \frac{1}{2} f''(r)\, e_n^2\,(e_n - e_{n-1})}{f'(r)(e_n - e_{n-1}) + \frac{1}{2} f''(r)\,(e_n^2 - e_{n-1}^2)}$$

$$= \frac{-\frac{1}{2}f''(r)e_n e_{n-1}^2 + \frac{1}{2}f''(r)e_n^2 e_{n-1}}{f'(r)(e_n - e_{n-1}) + \frac{1}{2}f''(r)(e_n^2 - e_{n-1}^2)}$$

expect super-linear convergence
$\Rightarrow e_n \ll e_{n-1}$

$$\approx \frac{-\frac{1}{2}f''(r)e_n e_{n-1}^2}{-f'(r)e_{n-1}} = \frac{f''(r)}{2f'(r)} e_n e_{n-1}$$

Assume $f''(r) \neq 0$, then $e_{n+1} \approx C e_n e_{n-1}$, $C \neq 0$

Suppose $|e_n| \approx A |e_{n-1}|^\alpha$, then $|e_{n+1}| \approx A(A|e_{n-1}|^\alpha)^\alpha$

$$= A^{\alpha+1} |e_{n-1}|^{\alpha^2}$$

$$A^{\alpha+1}|e_{n-1}|^{\alpha^2} \approx C A |e_{n-1}|^\alpha \cdot |e_{n-1}|$$

Equating powers of $|e_{n-1}|$, we get

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0 \qquad \alpha = \frac{1 \pm \sqrt{5}}{2} \rightsquigarrow \frac{1+\sqrt{5}}{2} \approx 1.618$$

(between linear and quadratic convergence).

- Compare Steffensen's and Secant (two-steps)

$$e_n \rightsquigarrow A e_n^2 \qquad e_n \rightsquigarrow B e_n^\alpha \rightarrow B(Be_n^\alpha)^\alpha = B^{\alpha+1} e_n^{\alpha^2}$$

$$\alpha^2 \approx 2.618$$

- Newton's has multi-D generalization, but Steffensen's and Secant don't have.

3.5 Computing zeros of polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_k \in \mathbb{C}, \ a_n \neq 0$$

Want to find zeros of $p(z)$.

Newton's, Steffensen's, Secant can be applied (but NOT bisection!)

Basic properties of polynomials

Thm (Fundamental thm of algebra) Any nonconstant complex polynomial has at least a complex zero.

- For any $c \in \mathbb{C}$, one can write $p(z)$ (w/ $\deg(p) = n$) as

$$p(z) = (z-c)q(z) + r$$

where $\deg(q) = n-1$ and $r \in \mathbb{C}$. If c is a zero of $p(z)$, then $r = 0$, i.e. $p(z) = (z-c)q(z)$.

- Using this, once you find one zero of $p(z)$, you can factor it out and deal w/ $q(z)$ w/ $\deg = n-1$.

- Any $p(z)$ can be written as

$$p(z) = a_n (z-r_1)^{m_1} \cdots (z-r_\ell)^{m_\ell}$$

where $r_1, \cdots, r_\ell \in \mathbb{C}$ are the zeros of $p(z)$ and $m_1, \cdots, m_\ell$ are multiplicities, satisfying $\sum_{k=1}^{\ell} m_k = n$

- Upper bound of zeros

Thm Any zero r of $p(z)$ satisfies $|r| \le \rho$ where

$$\rho = 1 + |a_n|^{-1} \max_{0 \le k < n} |a_k|$$

Proof If $|z| > \rho$, then

$$|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \le C\left(|z|^{n-1} + \cdots + |z| + 1\right) \qquad C := \max_{0 \le k < n} |a_k|$$

$$= C \frac{|z|^n - 1}{|z| - 1}$$

$$|a_n z^n| \ge |a_n| \cdot |z|^n$$

$$|p(z)| \ge |a_n z^n| - |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

$$\ge |a_n| \cdot |z|^n - C \frac{|z|^n - 1}{|z| - 1}$$

$$= \frac{|a_n| \cdot |z|^n (|z| - 1) - C(|z|^n - 1)}{|z| - 1} \qquad |z| - 1 > 0$$

$$= \frac{|a_n| \cdot |z|^n \left(|z| - 1 - C|a_n|^{-1} + C|a_n|^{-1} \cdot |z|^{-n}\right)}{|z| - 1} > 0$$

since $|z| > \rho = 1 + C|a_n|^{-1}$

Horner's algorithm

Given $p(z)$ and z_0, went to calculate $p(z_0)$ and $q(z) = \dfrac{p(z) - p(z_0)}{z - z_0}$

(i.e. $p(z) = (z - z_0) q(z) + p(z_0)$)

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = (z - z_0)(b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + \cdots + b_1 z + b_0) + p(z_0)$$

deg n : $b_{n-1} = a_n$

deg $n-1$: $b_{n-2} = b_{n-1} z_0 + a_{n-1}$

deg $n-2$: $b_{n-3} = b_{n-2} z_0 + a_{n-2}$

$\vdots$

deg 1 : $b_0 = b_1 z_0 + a_1$

deg 0 : $p(z_0) = b_0 z_0 + a_0$

Calculate iteratively

total cost: n multiplications, n additions.

(it is also the most effective way of calculating $p(z_0)$)