3.4 Fixed point and functional iteration

Recall Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x_n) \approx \frac{f(x_n+h) - f(x_n)}{h}$$

$$h = f(x_n)$$

avoid f'

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n+f(x_n)) - f(x_n)}$$ Steffensen's method

Both methods have the form

$$x_{n+1} = F(x_n), \quad n \geq 0$$ "functional iteration"

for some function F. Suppose $\{x_n\}$ converges to s, then (assuming F continuous)

$$s = F(s)$$

i.e., s is a <u>fixed point</u> of F.

For Newton's method, $F(x) = x - \dfrac{f(x)}{f'(x)}$

$$f(r) = 0 \iff F(r) = r$$

(similar for Steffenson's, $F(x) = \begin{cases} x - \dfrac{(f(x))^2}{f(x+f(x)) - f(x)} & f(x) \neq 0 \\ x & f(x) = 0 \end{cases}$

• Convergence of functional iterations.

Def Let C be a closed subset of $\mathbb{R}^N$. A mapping $F: C \to C$ is said to be contractive if $\exists\ 0 < \lambda < 1$ s.t.

$$\| F(x) - F(y) \| \leq \lambda \| x - y \| \quad \forall\ x, y \in C$$

Thm (contrative mapping thm). Let $F: C \to C$ be a contractive mapping defined on a closed subset $C \subset \mathbb{R}^N$. Then there exists a unique fixed point $s \in C$ of F. Furthermore, for any initial point $x_0 \in C$, the functional iteration $x_{n+1} = F(x_n)$ converges to s, w/ error estimate

$$\| x_{n+1} - s \| \leq \lambda \| x_n - s \| \qquad n \geq 0.$$

(at least linear convergence)

Proof Existence: take any $x_0 \in C$, and define $\{x_n\}$ by $x_{n+1} = F(x_n)$

$$\| x_{n+1} - x_n \| = \| F(x_n) - F(x_{n-1}) \| \leq \lambda \| x_n - x_{n-1} \|$$

$$\Rightarrow \quad \| x_{n+1} - x_n \| \leq \lambda^n \| x_1 - x_0 \|$$

Claim: $\{x_n\}$ is a Cauchy sequence

Proof: for $n > m > M$

$$\| x_n - x_m \| \leq \sum_{k=m}^{n-1} \| x_{k+1} - x_k \| \leq \sum_{k=m}^{n-1} \lambda^k \| x_1 - x_0 \|$$

$$\leq \sum_{k=m}^{\infty} \lambda^k \| x_1 - x_0 \| = \lambda^m \cdot \frac{1}{1-\lambda} \| x_1 - x_0 \|$$

$$\leq \lambda^M \cdot \frac{1}{1-\lambda} \| x_1 - x_0 \|$$

Therefore $s := \lim_{n \to \infty} x_n$ exists, and $s \in C$ because C is closed

Then, from $x_{n+1} = F(x_n)$, taking $n \to \infty$, using continuity of F
(from contractive)

we get $s = F(s)$.

<u>Uniqueness</u>: if $s_1, s_2 \in C$ are fixed points of F, then

$$\|s_1 - s_2\| = \|F(s_1) - F(s_2)\| \le \lambda \|s_1 - s_2\| \quad \Rightarrow \quad \|s_1 - s_2\| = 0$$
$$s_1 = s_2$$

<u>Error estimate</u>: for any initial $x_0 \in C$

$$\|x_{n+1} - s\| = \|F(x_n) - F(s)\| \le \lambda \|x_n - s\|$$

- Similar conclusion holds in Banach space.

- For $F \in C^1(\mathbb{R})$, if $|F'(x)| \le \lambda < 1 \quad \forall x \in \mathbb{R}$, then

  F is contractive on $\mathbb{R}$

  $$|F(x) - F(y)| = |F'(\xi)(x-y)| \le \lambda |x-y|$$

  <u>Ex</u> $F(x) = \frac{1}{2}x + \frac{1}{3}\sin x + 1$

  $$F'(x) = \frac{1}{2} + \frac{1}{3}\cos x \qquad |F'(x)| \le \frac{5}{6} \quad \Rightarrow \text{contractive on } \mathbb{R}$$

- For $F \in C^1(\mathbb{R})$, if s is a fixed point w/ $|F'(s)| < 1$

  then $\exists\, \delta > 0$ s.t. F maps $[s-\delta, s+\delta]$ into itself, and is contractive on $[s-\delta, s+\delta]$.

<u>Proof</u> Take δ small so that $|F'(x)| \le \lambda < 1 \quad \forall x \in [s-\delta, s+\delta]$

If $x \in [s-\delta, s+\delta]$, then

$$|F(x) - s| = |F(x) - F(s)| = |F'(\xi)(x-s)| \le \lambda\delta$$

$$\Rightarrow F(x) \in [s-\delta, s+\delta].$$

If $x, y \in [s-\delta, s+\delta]$, then

$$|F(x) - F(y)| = |F'(\eta)(x-y)| \leq \lambda |x-y|$$

$\underset{\sim}{\text{Ex}}$ Newton's : $F(x) = x - \dfrac{f(x)}{f'(x)}$ Suppose r is a simple root of f

$$F'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{(f'(x))^2}$$

$$F'(r) = 0$$

Convergence rate for functional iteration (heuristic)

Let s be a fixed point of F. Define $e_n = x_n - s$

Suppose $F^{(k)}(s) = 0$, $k = 1, \cdots, q-1$, $F^{(q)}(s) \neq 0$

$$x_{n+1} = F(x_n)$$

$$\begin{aligned} e_{n+1} &= F(x_n) - s = F(x_n) - F(s) \\ &= F'(s) e_n + \frac{1}{2!} F''(s) e_n^2 + \cdots + \underset{\sim}{\frac{1}{q!}} F^{(q)}(\xi) e_n^q \\ &= \underbrace{\frac{1}{q!} F^{(q)}(\xi)}_{\text{expect nonzero}} e_n^q \end{aligned}$$

- If $q = 1$, then
  - If $|F'(s)| < 1$, expect linear convergence w/ $\lambda \approx |F'(s)|$
  - If $|F'(s)| > 1$, expect no convergence !
- If $q \gtrsim 2$, then expect order-q convergence

Ex Newton's $F(x) = x - \dfrac{f(x)}{f'(x)}$ $\qquad$ r : Simple root

$$F'(x) = \frac{f(x)\, f''(x)}{(f'(x))^2} \qquad F'(r) = 0$$

$$F''(x) = \frac{\left(f'(x)f''(x) + f(x)f'''(x)\right)\cdot(f'(x))^2 - f(x)f''(x)\cdot 2f'(x)f''(x)}{(f'(x))^4}$$

$$F''(r) = \frac{f''(r)}{f'(r)}$$

$\begin{cases} \text{If } f''(r) \neq 0 \rightsquigarrow \text{quadratic convergence} \\ \text{If } f''(r) = 0 \longrightarrow \text{at least cubic convergence.} \\ \text{(inflection point)} \end{cases}$