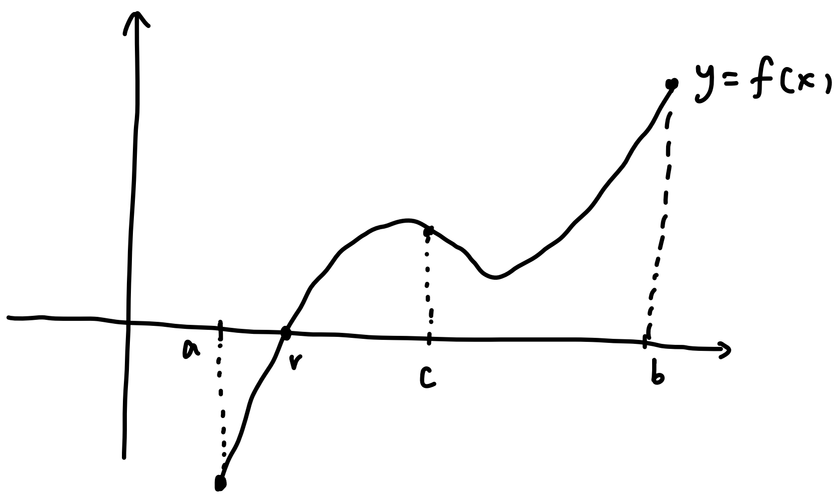
3.1 Bisection method

- If $f \in C[a, b]$ w/ $f(a)f(b) < 0$, then the intermediate value theorem tells us that

$$\exists\ r \in (a, b) \quad \text{s.t.} \quad f(r) = 0$$

Problem: Find such a root r.



Say, $f(a) < 0$, $f(b) > 0$

① Take $c = \dfrac{a+b}{2}$

② If $f(c) = 0$ done

If $f(c) > 0$, then $\exists$ root in $[a, c]$

If $f(c) < 0$, then $\exists$ root in $[c, b]$

③ update the interval into $[a, c]$ or $[c, b]$ and iterate.

- Write a pseudocode

Input: $a < b$, $f \in C[a, b]$ w/ $f(a)f(b) < 0$, TOL, M

Output: c as an approximation of a root of f in $[a, b]$

(a_n, b_n s.t. $[a_n, b_n]$ contains a root)

$u = f(a)$

$v = f(b)$

for $i = 1 : M$

$\quad c = \dfrac{a+b}{2}$ $\qquad \left(a + \dfrac{b-a}{2} \right)$

$\quad w = f(c)$

$\quad$ If $\text{sign}(u) \neq \text{sign}(w)$ then

$\qquad b = c; \quad v = w$

$\quad$ else

$\qquad a = c; \quad u = w$

$\quad$ end if

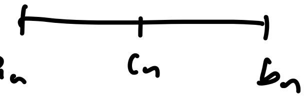
$\quad$ If $b - a < TOL$ then break

end

output a, b

- If we stop at n-th iteration and $[a_n, b_n]$, to output an approximation of the root, then we output

$$c_n = \frac{a_n + b_n}{2}$$



Error analysis

Suppose we start from $[a_0, b_0]$, then

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \qquad n = 0, 1, 2, \ldots$$

$$b_n - a_n = 2^{-n}(b_0 - a_0) \longrightarrow 0 \quad \text{as } n \to \infty$$

Also, $\{a_n\}$ is increasing, $\{b_n\}$ is decreasing

$\Rightarrow \lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exists, and are equal. Call it r.

Since $f(a_n) f(b_n) \le 0$, taking $n \to \infty$, using $f \in C[a,b]$.

we get $f(r)^2 \le 0 \Rightarrow f(r) = 0$ (r is a root of f)

Also, by sandwich thm, $\lim_{n\to\infty} c_n = r$.

Thm $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$, $\lim_{n\to\infty} c_n$ exist and are equal. Call it r,

then r is a root of f. $\{c_n\}$ satisfies

$$|r - c_n| \le 2^{-(n+1)} (b_0 - a_0)$$

In particular, $\lim_{n\to\infty} c_n = r$

(textbook sec. 1.2)

If x^* is a desired sol'n to a problem, and an iterative numerical method produces a sequence of approximations $\{x_n\}$ satisfying $\lim_{n\to\infty} x_n = x^*$ (in certain sense)

then we say this numerical method converges

- How fast is the convergence?

Def if $|x_{n+1} - x^*| \le \lambda |x_n - x^*|$ $\forall n \ge N$ for some $N \in \mathbb{Z}$ and $0 \le \lambda < 1$, then we say this numerical method's rate of convergence is at least linear

- Linear convergence implies $|x_n - x^*| \le C \lambda^n$ $\forall n \ge N$ for some $C > 0$.

Therefore bisection converges as fast as linear convergence w/ $\lambda = \frac{1}{2}$

- Generally, if $|x_{n+1} - x^*| \le C\,|x_n - x^*|^{\alpha}$ $\forall\, n \ge N$ for some $C > 0$

and $\alpha > 1$, then the rate of convergence is

of order α (at least)

$$C\,|x_n - x^*|^{\alpha} = C\,|x_n - x^*|^{\alpha-1} \cdot |x_n - x^*|$$

$\alpha = 2$: quadratic ; $\alpha = 3$: cubic.

$\hookrightarrow$ 10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}

Evaluate bisection method:

- Convergence rate : linear convergence (almost), decent but not good enough.
- Stability : good.
- Regularity / information : good. only need continuity/point values
- Multi-dim : Cannot be done directly.

- More than one root

bisection (many other methods) can only find one root.