Total time: 10 minutes.

Problem 1 (10 points). Let

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 3 \\ -1 \end{pmatrix} \right\}$$

be a subset of $\mathbb{R}^4$. Find a basis for the orthogonal complement of S.

Denote

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 3 \\ 2 & -1 \end{pmatrix}$$

Then the orthogonal complement of S is $(\text{Col } A)^\perp = \text{Nul } (A^T)$. To find a basis for $\text{Nul } (A^T)$, do row reductions as

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & -4 & 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \underline{1} & 2 & 0 & 5 \\ 0 & 0 & \underline{1} & 3 \end{pmatrix}$$

The solutions to $A^T\mathbf{x} = 0$ are

$$\begin{cases} x_1 = -2x_2 - 5x_4 \\ x_2 \text{ free} \\ x_3 = -3x_4 \\ x_4 \text{ free} \end{cases}, \quad \mathbf{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

Therefore a basis for the orthogonal complement of S is

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$